

# ON THE MICROSCOPIC SPACETIME CONVEXITY PRINCIPLE OF FULLY NONLINEAR PARABOLIC EQUATIONS II: SPACETIME QUASICONCAVE SOLUTIONS

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**ABSTRACT.** Spacetime convexity is a basic geometric property of the solutions of parabolic equations. In this paper, we establish the microscopic spacetime convexity principle of spacetime quasiconcave solutions of fully nonlinear parabolic equations under a general structural condition. As an application, we get a global geometric lower bound estimate of the principal curvature of spatial level sets of spacetime quasiconcave solutions in convex rings. In fact, the results holds for the  $p$ -Laplacian parabolic equations ( $p > 1$ ) and mean curvature parabolic equations.

## 1. INTRODUCTION

This is the second paper devoted to the microscopic spacetime convexity principle of the second fundamental forms of the spatial and spacetime level sets of the solutions to fully nonlinear parabolic equations. Meanwhile, we get a positive lower bound estimate of the principal curvature of spatial level sets of spacetime quasiconcave solutions in convex rings.

Spacetime convexity is a basic geometric property of the solutions of parabolic equations. In [5, 6, 7], Borell used the Brownian motion to study certain spacetime convexities of the solutions of diffusion equations and the level sets of the solution to heat equations with Schrödinger potential. In this paper, we consider the spatial convexity and the spacetime convexity of the level sets of the spacetime quasiconcave solutions to heat equation. A continuous function  $u(x, t)$  on  $\Omega \times (0, T]$  is called *spacetime quasiconcave* if the spacetime superlevel sets  $\{(x, t) \in \Omega \times (0, T) | u(x, t) \geq c\}$  are convex for each constant  $c$ .

The convexity of the level sets of the solutions to elliptic partial differential equations has been studied extensively. For instance, Ahlfors [1] contains the well-known result that level curves of Green function on simply connected convex domain in the plane are the convex Jordan curves. In 1956, Shiffman [27] studied the minimal annulus in  $\mathbb{R}^3$  whose boundary consists of two closed convex curves in parallel planes  $P_1, P_2$ . He proved that the intersection of the surface with any parallel plane  $P$ , between  $P_1$  and  $P_2$ , is a convex Jordan curve. In 1957, Gabriel [14] proved that the level sets of the Green function on a 3-dimensional bounded convex domain are strictly convex. In 1977, Lewis [20] extended Gabriel's result to  $p$ -harmonic functions in higher dimensions. Caffarelli-Spruck [9] generalized the Lewis [20] results to a class of semilinear elliptic partial differential equations. Motivated by the result of Caffarelli-Friedman [8], Korevaar [18] gave a new proof on the results of Gabriel and Lewis by applying the deformation process and the constant rank theorem of the second fundamental form of the convex level sets of  $p$ -harmonic function.

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A survey of this subject is given by Kawohl [17]. For more recent related extensions, please see the papers by Bianchini-Longinetti-Salani [4], Bian-Guan [2], Xu [31] and Bian-Guan-Ma-Xu [3].

There is also an extensive literature on the curvature estimates of the level sets of the solutions to elliptic partial differential equations. For 2-dimensional harmonic function and minimal surface with convex level curves, Ortel-Schneider [26], Longinetti [22] and [23] proved that the curvature of the level curves attains its minimum on the boundary (see Talenti [28] for related results). Longinetti also studied the precise relation between the curvature of the convex level curves and the height of 2-dimensional minimal surface in [23]. Ma-Ou-Zhang [24] got the Gaussian curvature estimates of the convex level sets on higher dimensional harmonic function, and Wang-Zhang [30] got the similar curvature estimates of some quasi-linear elliptic equations under certain structure condition [4]. Both of their test functions involved the Gaussian curvature of the boundary and the norm of the gradient on the boundary. Furthermore, for the  $p$ -harmonic function with strictly convex level sets, Ma-Zhang [25] obtained that the curvature function introduced in it is concave with respect to the height of the  $p$ -harmonic function. For the principal curvature estimates in higher dimension, in terms of the principal curvature of the boundary and the norm of the gradient on the boundary, Chang-Ma-Yang [10] obtained the lower bound estimates of principal curvature for the strictly convex level sets of higher dimensional harmonic functions and solutions to a class of semilinear elliptic equations under certain structure condition [4]. Recently, in Guan-Xu [16], they got a lower bound for the principal curvature of the level sets of solutions to a class of fully nonlinear elliptic equations in convex rings under the general structure condition [4] via the approach of constant rank theorem.

Let us introduce some notation.

**Definition 1.1.** For each  $\theta \in \mathbb{S}^{n-1}$ , denote  $\theta^\perp$  the linear subspace in  $\mathbb{R}^n$  which is orthogonal to  $\theta$ . Define  $\mathcal{S}_n^-(\theta)$  to be the class of  $n \times n$  symmetric real matrices which are negative definite on  $\theta^\perp$ . Denote  $\mathcal{S}_n^{0-}(\theta)$  the subclass of  $\mathcal{S}_n^-(\theta)$  of matrices that have  $\theta$  as eigenvector with corresponding null eigenvalue. For any  $b \in \mathbb{R}^n$  with  $s = \langle b, \theta \rangle > 0$ , define

$$\mathcal{B}_\theta^-(Y) = \left\{ B \in \mathcal{S}^{n+1} \mid B = \begin{pmatrix} \bar{B} & b^T \\ b & \chi \end{pmatrix} \text{ with } \bar{B} \in \mathcal{S}_n^{0-}(\theta) \cap Y, \chi \in \mathbb{R} \right\},$$

where  $\mathcal{S}^{n+1}$  denote the space of real symmetric  $n+1 \times n+1$  matrices,

Denote  $J = (I_n | 0)$  the  $n \times (n+1)$  matrix, where  $I_n$  is the  $n \times n$  identity matrix and 0 is the null vector in  $\mathbb{R}^n$ .

In this paper, we consider the spacetime quasiconcave solution to fully nonlinear parabolic equation as follows,

$$(1.1) \quad \frac{\partial u}{\partial t} = F(\nabla^2 u, \nabla u, u, x, t), \text{ in } \Omega \times (0, T],$$

where  $\Omega$  is a domain in  $\mathbb{R}^n$ ,  $\nabla u = (u_{x_1}, \dots, u_{x_n})$  and  $\nabla^2 u = (u_{x_i x_j})_{n \times n}$ . Let  $\Lambda \subset \mathcal{S}^n$  be an open set,  $\mathbb{S}^{n-1}$  a unit sphere and  $F = F(r, p, u, x, t)$  a  $C^{2,1}$  function in  $\Lambda \times \mathbb{R}^n \times \mathbb{R} \times \Omega \times [0, T]$ . We will assume that  $F$  satisfies the following conditions:

$$(1.2) \quad (F^{\alpha\beta}) := \left( \frac{\partial F}{\partial r_{\alpha\beta}}(\nabla^2 u, \nabla u, u, x, t) \right) > 0, \quad \forall (x, t) \in \Omega \times [0, T],$$

and for each  $(\theta, u) \in \mathbb{S}^{n-1} \times \mathbb{R}$  fixed,

$$(1.3) \quad F(s^{-1}JB^{-1}J^T, s^{-1}\theta, u, x, t) \text{ is locally concave in } (B, x, t).$$

In fact, we always assume

$$(1.4) \quad |\nabla u| > 0 \text{ and } u_t > 0 \text{ in } \Omega \times (0, T].$$

Now we state our theorems.

**Theorem 1.2.** *Suppose  $u \in C^{3,1}(\Omega \times (0, T])$  is a spacetime quasiconcave to fully nonlinear parabolic equation (1.1), and  $F$  satisfies conditions (1.2)-(1.4). Then the second fundamental form of spacetime level sets  $\hat{\Sigma}^c = \{(x, t) \in \Omega \times (0, T) | u(x, t) = c\}$  has the same constant rank in  $\Omega$  for each fixed  $t \in (0, T]$ . Moreover, let  $l(t)$  be the minimal rank of the second fundamental form in  $\Omega$ , then  $l(s) \leq l(t)$  for all  $0 < s \leq t \leq T$ .*

For the study of the spacetime level sets of fully nonlinear equation, we should consider the spatial level sets first. Suppose  $u$  is the spacetime quasiconcave solution to fully nonlinear parabolic equation (1.1), then  $u$  is also spatial quasiconcave, that is the spatial level sets  $\Sigma_x^{c,t} = \{x \in \Omega | u(x, t) = c\}$  are all convex. And we get the following constant rank theorem for the second fundamental form of the spatial level sets.

**Theorem 1.3.** *Suppose  $u \in C^{3,1}(\Omega \times (0, T])$  is a spacetime quasiconcave to fully nonlinear parabolic equation (1.1), and  $F$  satisfies conditions (1.2)-(1.4). Then the second fundamental form of spatial level sets  $\Sigma^c = \{x \in \Omega | u(x, t) = c\}$  has the same constant rank in  $\Omega$  for each fixed  $t \in (0, T]$ . Moreover, let  $l(t)$  be the minimal rank of the second fundamental form in  $\Omega$ , then  $l(s) \leq l(t)$  for all  $0 < s \leq t \leq T$ .*

**Remark 1.4.** *In fact, in the proof of Theorem 1.3, we just need a weaker structural condition as follows instead of (1.3),*

$$(1.5) \quad F(s^{-1}JB^{-1}J^T, s^{-1}\theta, u, x, t) \text{ is locally concave in } (B, x), \quad \text{for fixed } (\theta, u) \in \mathbb{S}^{n-1} \times \mathbb{R}.$$

*But the condition that  $u$  is spacetime quasiconcave is necessary, and it is the main difference between Theorem 1.3 and the result in [12]. That is, if  $u$  is spacetime quasiconcave, the constant rank theorem for spatial level sets holds for the parabolic equations with (1.5). Otherwise, if  $u$  is spatial quasiconcave, the constant rank theorem for spatial level sets holds for the parabolic equations with the totally different structural condition in [12].*

Inspired by [16], we also consider to establish a geometric lower bound for the principal curvature of the spatial level sets of solutions to parabolic equations in the convex rings as follows,

$$(1.6) \quad \begin{cases} \frac{\partial u}{\partial t} = F(\nabla^2 u, \nabla u, u, x, t) & \text{in } \Omega \times (0, T], \\ u(x, 0) = u_0(x) & \text{in } \Omega, \\ u(x, t) = 0 & \text{on } \partial\Omega_0 \times (0, T], \\ u(x, t) = 1 & \text{on } \partial\Omega_1 \times (0, T], \end{cases}$$

where  $\Omega = \Omega_0 \setminus \overline{\Omega_1}$ ,  $\Omega_0, \Omega_1$  are two convex domains with  $\overline{\Omega_1} \subset \Omega_0$ . Also we assume  $u_0$  is quasiconcave and satisfies

$$(1.7) \quad \begin{cases} F(\nabla^2 u_0, \nabla u_0, u_0, x, 0) > 0 & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega_0, \\ u_0 = 1 & \text{on } \partial\Omega_1. \end{cases}$$

We denote  $\kappa_s(x, t)$  the smallest principal curvature of the spatial level set  $\Sigma^{u(x_0, t)} = \{x \in \Omega | u(x, t) = u(x_0, t)\}$  at  $(x, t)$ . For each  $(x_0, t)$ , set

$$(1.8) \quad \kappa^{u(x_0, t)} = \inf_{x \in \Sigma^{u(x_0, t)}} \kappa_s(x, t).$$

We will assume that there exists  $\lambda > 0$  and  $\sigma > 0$ , such that

$$(1.9) \quad (F^{\alpha\beta}(\nabla^2 u, \nabla u, u, x, t)) \geq \lambda I_n, \quad \forall (x, t) \in \Omega \times (0, T].$$

and

$$(1.10) \quad |\nabla u| \geq \sigma, \text{ and } u_t \geq \sigma \quad \text{in } \Omega \times (0, T].$$

**Theorem 1.5.** *Suppose  $u \in C^{3,1}(\Omega \times (0, T])$  is a spacetime quasiconcave solution to fully nonlinear parabolic equation (1.6) and satisfies (1.10), and  $F$  satisfies conditions (1.7) and (1.9). Then*

$$(1.11) \quad \kappa^{u(x, t)} \geq \min\{\kappa^0, \kappa^1 e^{-A}\} e^{Au(x, t)} \quad \text{in } \Omega \times [0, T],$$

for some universal constant  $A$  depending only on  $\|F\|_{C^2}$ ,  $n$ ,  $\lambda$ ,  $\sigma$ ,  $\|u\|_{C^3}$ .

**Remark 1.6.** *The structural condition (1.3) and (1.5) is not optimal. There are some additional constrains from the equation and the spacetime convexity of solutions.*

In fact, the  $p$ -Laplacian operator and the mean curvature operators, that is, the parabolic equations

$$(1.12) \quad u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u), \quad p > 1,$$

$$(1.13) \quad u_t = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right),$$

$$(1.14) \quad u_t = (1 + |\nabla u|^2)^{\frac{3}{2}} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right),$$

do not satisfy the structure condition (1.4) or (1.6). But we have the following theorem.

**Theorem 1.7.** (1) *Theorem 1.2 and Theorem 1.3 holds for the spacetime quasiconcave solutions to the parabolic equations (1.12), (1.13) and (1.14) with the assumption (1.4).*

(2) *Theorem 1.5 holds for the spacetime quasiconcave solutions to the parabolic equations (1.12), (1.13) and (1.14) with the assumption (1.10).*

**Remark 1.8.** *Theorem 1.3 and Theorem 1.5 may be looked as some parabolic versions for Theorem 1.1 in [3] and Theorem 1.5 in [16] respectively. Also, the results are due to [12].*

The rest of the paper is organized as follows. In Section 2, we do some preliminaries. In Section 3, we prove the constant rank theorem of the spatial second fundamental form, that is Theorem 1.3 and part of (1) of Theorem 1.7. For the constant rank theorem of the spacetime fundamental form is proved in Section 4, including Theorem 1.2 and part of (1) of Theorem 1.7. As an application, the curvature estimate of Theorem 1.5 is proved in Section 5, and the left part of Theorem 1.7 is verified similarly. At last, we give some discussions.

## 2. PRELIMINARIES

In this section, we will give some preliminaries.

First, we introduce the definitions of spatial quasiconcave and spacetime quasiconcave.

**Definition 2.1.** A continuous function  $u(x, t)$  on  $\Omega \times (0, T]$  is called *spatial quasiconcave* if its superlevel sets  $\{x \in \Omega | u(x, t) \geq c\}$  are convex for each constant  $c$  and any fixed  $t \in (0, T]$ . And  $u(x, t)$  is called *spacetime quasiconcave* if its superlevel sets  $\{(x, t) \in \Omega \times (0, T) | u(x, t) \geq c\}$  are convex for each constant  $c$ .

In the following, we always assume  $\nabla u = (u_1, \dots, u_n)$  is the spatial gradient of  $u$  and  $Du = (u_1, \dots, u_n, u_t)$  the spacetime gradient.

**2.1. Spatial level sets and the spatial second fundamental form.** Suppose  $u(x, t) \in C^{2,1}(\Omega \times (0, T])$ , and  $u_n \neq 0$  for any fixed  $(x, t) \in \Omega \times (0, T]$ . It follows that the upward inner normal direction of the spatial level sets  $\Sigma^c = \{x \in \Omega | u(x, t) = c\}$  is

$$(2.1) \quad \vec{\nu} = \frac{|u_n|}{|\nabla u| u_n} (u_1, u_2, \dots, u_{n-1}, u_n),$$

where  $\nabla u = (u_1, u_2, \dots, u_{n-1}, u_n)$  is the spatial gradient of  $u$ .

The second fundamental form  $II$  of the spatial level sets of function  $u$  with respect to the upward normal direction (2.1) is

$$b_{ij} = -\frac{|u_n|(u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn})}{|\nabla u| u_n^3}, \quad 1 \leq i, j \leq n-1.$$

Set

$$(2.2) \quad h_{ij} = u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn}, \quad 1 \leq i, j \leq n-1,$$

then we may write

$$b_{ij} = -\frac{|u_n| h_{ij}}{|\nabla u| u_n^3}.$$

Note that if  $\Sigma^c = \{x \in \Omega | u(x, t) = c\}$  is locally convex, then the second fundamental form of  $\Sigma^c$  is semi-positive definite with respect to the upward normal direction (2.1). Let  $a(x, t) = (a_{ij}(x, t))$  be the symmetric Weingarten tensor of  $\Sigma^c = \{x \in \Omega | u(x, t) = c\}$ , then  $a$  is semipositive definite. As computed in [3], if  $u_n \neq 0$ , and the Weingarten tensor is

$$(2.3) \quad a_{ij} = -\frac{|u_n|}{|\nabla u| u_n^3} A_{ij}, \quad 1 \leq i, j \leq n-1,$$

where

$$(2.4) \quad A_{ij} = h_{ij} - \frac{u_i u_j h_{ll}}{W(1+W)u_n^2} - \frac{u_j u_l h_{il}}{W(1+W)u_n^2} + \frac{u_i u_j u_k u_l h_{kl}}{W^2(1+W)^2 u_n^4}, \quad W = \frac{|\nabla u|}{|u_n|}.$$

With the above notations, at the point  $(x, t)$  where  $u_n(x, t) = |\nabla u(x, t)| > 0$ ,  $u_i(x, t) = 0$ ,  $i = 1, \dots, n-1$ ,  $a_{ijk}$  is commutative, that is, they satisfy the Codazzi property  $a_{ij,k} = a_{ik,j}$ ,  $\forall i, j, k \leq n-1$ .

**2.2. Spacetime level sets and the spacetime second fundamental form.** Suppose  $u(x, t) \in C^{2,1}(\Omega \times (0, T])$ , and  $u_t \neq 0$  for any fixed  $(x, t) \in \Omega \times (0, T]$ . It follows that the upward inner normal direction of the spatial level sets  $\hat{\Sigma}^c = \{(x, t) \in \Omega \times (0, T) | u(x, t) = c\}$  is

$$(2.5) \quad \vec{\nu} = \frac{|u_t|}{|Du|u_t}(u_1, u_2, \dots, u_{n-1}, u_n, u_t),$$

where  $Du = (u_1, u_2, \dots, u_{n-1}, u_n, u_t)$  is the spacetime gradient of  $u$ .

The second fundamental form  $II$  of the spacetime level sets of function  $u$  with respect to the upward normal direction (2.5) is

$$\hat{b}_{\alpha\beta} = -\frac{|u_t|(u_t^2 u_{\alpha\beta} + u_{tt} u_\alpha u_\beta - u_t u_\beta u_{\alpha t} - u_t u_\alpha u_{\beta t})}{|Du|u_t^3}, \quad 1 \leq \alpha, \beta \leq n.$$

Set

$$(2.6) \quad \hat{h}_{\alpha\beta} = u_t^2 u_{\alpha\beta} + u_{tt} u_\alpha u_\beta - u_t u_\beta u_{\alpha t} - u_t u_\alpha u_{\beta t}, \quad 1 \leq \alpha, \beta \leq n,$$

then we may write

$$\hat{b}_{\alpha\beta} = -\frac{|u_t|\hat{h}_{\alpha\beta}}{|Du|u_t^3}.$$

Note that if  $\hat{\Sigma}^c = \{(x, t) \in \Omega \times (0, T) | u(x, t) = c\}$  is locally convex, then the second fundamental form of  $\Sigma^{c,t}$  is semipositive definite with respect to the upward normal direction (2.5). Let  $\hat{a}(x, t) = (\hat{a}_{ij}(x, t))$  be the symmetric Weingarten tensor of  $\hat{\Sigma}^c = \{(x, t) \in \Omega \times (0, T) | u(x, t) = c\}$ , then  $\hat{a}$  is semipositive definite. If  $u_t \neq 0$ , and the Weingarten tensor is

$$(2.7) \quad \hat{a}_{\alpha\beta} = -\frac{|u_t|}{|Du|u_t^3} \hat{A}_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n,$$

where

$$(2.8) \quad \hat{A}_{\alpha\beta} = \hat{h}_{\alpha\beta} - \frac{u_\alpha u_\gamma \hat{h}_{\beta\gamma}}{\hat{W}(1 + \hat{W})u_t^2} - \frac{u_\beta u_\gamma \hat{h}_{\alpha\gamma}}{\hat{W}(1 + \hat{W})u_t^2} + \frac{u_\alpha u_\beta u_\gamma u_\eta \hat{h}_{\gamma\eta}}{\hat{W}^2(1 + \hat{W})^2 u_t^4}, \quad \hat{W} = \frac{|Du|}{|u_t|}.$$

With the above notations, at the point  $(x, t)$  where  $u_t(x, t) > 0$ ,  $u_n(x, t) = |\nabla u(x, t)| > 0$ ,  $u_i(x, t) = 0$ ,  $i = 1, \dots, n-1$ , we get

$$(2.9) \quad 1 - \frac{u_n^2}{\hat{W}(1 + \hat{W})u_t^2} \equiv \frac{\hat{W}u_t^2 + \hat{W}^2 u_t^2 - u_n^2}{\hat{W}(1 + \hat{W})u_t^2} = \frac{1}{\hat{W}}.$$

So

$$(2.10) \quad \hat{A}_{\alpha\beta} = \hat{h}_{\alpha\beta} = u_t^2 u_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n-1;$$

$$(2.11) \quad \hat{A}_{\alpha n} = \hat{h}_{\alpha n} - \frac{u_n^2 \hat{h}_{\alpha n}}{\hat{W}(1 + \hat{W})u_t^2} = \frac{1}{\hat{W}} \hat{h}_{\alpha n} = \frac{1}{\hat{W}} [u_t^2 u_{\alpha n} - u_t u_n u_{\alpha t}], \quad 1 \leq \alpha \leq n-1;$$

$$(2.12) \quad \begin{aligned} \hat{A}_{nn} &= \hat{h}_{nn} - 2 \frac{u_n^2 \hat{h}_{nn}}{\hat{W}(1 + \hat{W})u_t^2} + \frac{u_n^4 \hat{h}_{nn}}{\hat{W}^2(1 + \hat{W})^2 u_t^4} = \frac{1}{\hat{W}^2} \hat{h}_{nn} \\ &= \frac{1}{\hat{W}^2} [u_t^2 u_{nn} + u_n^2 u_{tt} - 2u_t u_n u_{nt}]. \end{aligned}$$

Also, at any point  $(x, t)$ , we can translate the spacetime coordinate systems. When we choose the coordinates  $y = (y_1, \dots, y_n, y_{n+1})$  as a new spacetime coordinates, such that  $u_{y_{n+1}} > 0$ , the Weingarten tensor is

$$(2.13) \quad \bar{a}_{\alpha\beta} = -\frac{|u_{y_{n+1}}|}{|Du|u_{y_{n+1}}^3} \bar{A}_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n,$$

where

$$(2.14) \quad \bar{A}_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{u_{y_\alpha} u_{y_\gamma} \bar{h}_{\beta\gamma}}{\bar{W}(1 + \bar{W})u_{y_{n+1}}^2} - \frac{u_{y_\beta} u_{y_\gamma} \bar{h}_{\alpha\gamma}}{\bar{W}(1 + \bar{W})u_{y_{n+1}}^2} + \frac{u_{y_\alpha} u_{y_\beta} u_{y_\gamma} u_{y_\eta} \bar{h}_{\gamma\eta}}{\bar{W}^2(1 + \bar{W})^2 u_{y_{n+1}}^4}, \quad \bar{W} = \frac{|Du|}{|u_{y_{n+1}}|},$$

$$(2.15) \quad \bar{h}_{\alpha\beta} = u_{y_{n+1}}^2 u_{y_\alpha y_\beta} + u_{y_{n+1} y_{n+1}} u_{y_\alpha} u_{y_\beta} - u_{y_{n+1}} u_{y_\beta} u_{y_\alpha y_{n+1}} - u_{y_{n+1}} u_{y_\alpha} u_{y_\beta y_{n+1}}, \quad 1 \leq \alpha, \beta \leq n.$$

With the above notations, at the point  $(x, t)$  with the new coordinates  $y$  such that  $u_{y_i} = 0$  for any  $1 \leq i \leq n$  and  $u_{y_{n+1}} = |Du| > 0$ , we get

$$(2.16) \quad \bar{A}_{\alpha\beta} = \bar{h}_{\alpha\beta} = u_{y_{n+1}}^2 u_{y_\alpha y_\beta}, \quad 1 \leq \alpha, \beta \leq n,$$

**2.3. Elementary symmetric functions.** In this subsection, we recall the definition and some basic properties of elementary symmetric functions, which could be found in [15, 21].

**Definition 2.2.** For any  $k = 1, 2, \dots, n$ , we set

$$(2.17) \quad \sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}, \quad \text{for any } \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n.$$

We also set  $\sigma_0 = 1$  and  $\sigma_k = 0$  for  $k > n$ .

We denote by  $\sigma_k(\lambda|i)$  the symmetric function with  $\lambda_i = 0$  and  $\sigma_k(\lambda|ij)$  the symmetric function with  $\lambda_i = \lambda_j = 0$ .

We need the following standard formulas of elementary symmetric functions.

**Proposition 2.3.** Let  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$  and  $k = 0, 1, \dots, n$ , then

$$\begin{aligned} \sigma_k(\lambda) &= \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \quad \forall 1 \leq i \leq n, \\ \sum_{i=1}^n \lambda_i \sigma_{k-1}(\lambda|i) &= k \sigma_k(\lambda), \\ \sum_{i=1}^n \sigma_k(\lambda|i) &= (n-k) \sigma_k(\lambda). \end{aligned}$$

The definition can be extended to symmetric matrices by letting  $\sigma_k(W) = \sigma_k(\lambda(W))$ , where  $\lambda(W) = (\lambda_1(W), \lambda_2(W), \dots, \lambda_n(W))$  are the eigenvalues of the symmetric matrix  $W$ . We also denote by  $\sigma_k(W|i)$  the symmetric function with  $W$  deleting the  $i$ -row and  $i$ -column and  $\sigma_k(W|ij)$  the symmetric function with  $W$  deleting the  $i, j$ -rows and  $i, j$ -columns. Then we have the following identities.

**Proposition 2.4.** Suppose  $W = (W_{ij})$  is diagonal, and  $m$  is a positive integer, then

$$(2.18) \quad \frac{\partial \sigma_m(W)}{\partial W_{ij}} = \begin{cases} \sigma_{m-1}(W|i), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

and

$$(2.19) \quad \frac{\partial^2 \sigma_m(W)}{\partial W_{ij} \partial W_{kl}} = \begin{cases} \sigma_{m-2}(W|lk), & \text{if } i = j, k = l, i \neq k, \\ -\sigma_{m-2}(W|lk), & \text{if } i = l, j = k, i \neq j, \\ 0, & \text{otherwise.} \end{cases}$$

To study the rank of the spacetime second fundamental form  $\hat{a}$ , we need the following simple lemma.

**Lemma 2.5.** Suppose  $\hat{a} \geq 0$ , and  $l = \text{Rank}\{\hat{a}(x_0, t_0)\} \leq n - 1$ , and  $\left(\hat{a}_{\alpha\beta}(x_0, t_0)\right)_{n-1 \times n-1}$  is diagonal with  $\hat{a}_{11} \geq \hat{a}_{22} \geq \cdots \geq \hat{a}_{n-1n-1}$ , then at  $(x_0, t_0)$ , there is a positive constant  $C_0$  such that

CASE 1:

$$\begin{aligned} \hat{a}_{11} &\geq \cdots \geq \hat{a}_{l-1l-1} \geq C_0, \quad \hat{a}_{ll} = \cdots = \hat{a}_{n-1n-1} = 0, \\ \hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{in}^2}{\hat{a}_{ii}} &\geq C_0, \quad \hat{a}_{in} = 0, \quad l \leq i \leq n-1. \end{aligned}$$

CASE 2:

$$\begin{aligned} \hat{a}_{11} &\geq \cdots \geq \hat{a}_{ll} \geq C_0, \quad \hat{a}_{l+1l+1} = \cdots = \hat{a}_{n-1n-1} = 0, \\ \hat{a}_{nn} &= \sum_{i=1}^l \frac{\hat{a}_{in}^2}{\hat{a}_{ii}}, \quad \hat{a}_{in} = 0, \quad l+1 \leq i \leq n-1. \end{aligned}$$

PROOF. Set  $M = \left(\hat{a}_{\alpha\beta}(x_0, t_0)\right)_{n-1 \times n-1} = \text{diag}(\hat{a}_{11}, \hat{a}_{22}, \cdots, \hat{a}_{n-1n-1}) \geq 0$  and we can assume  $\text{Rank}\{M\} = k$  at  $(x_0, t_0)$ , then we can obtain  $k = l - 1$  or  $k = l$ . Otherwise, if  $k < l - 1$ , we know

$$\hat{a}_{l-1l-1} = \cdots = \hat{a}_{n-1n-1} = 0 \quad \text{at } (x_0, t_0),$$

and from  $\hat{a}(x_0, t_0) \geq 0$ , we get

$$\hat{a}_{l-1n} = \cdots = \hat{a}_{n-1n} = 0 \quad \text{at } (x_0, t_0).$$

So  $\text{Rank}\{\hat{a}\} \leq l - 1$ , contradiction. If  $k > l$ , we have

$$l = \text{Rank}\{\hat{a}\} \geq \text{Rank}\{M\} = k \geq l + 1 \quad \text{at } (x_0, t_0).$$

This is impossible.

For  $k = l - 1$ , we know at  $(x_0, t_0)$

$$\hat{a}_{11} \geq \cdots \geq \hat{a}_{l-1l-1} > 0, \quad \hat{a}_{ll} = \cdots = \hat{a}_{n-1n-1} = 0,$$

and due to  $\hat{a}(x_0, t_0) \geq 0$ , we get

$$\hat{a}_{ln} = \cdots = \hat{a}_{n-1n} = 0.$$

Since  $\text{Rank}\{\hat{a}\} = l$ , then  $\sigma_l(\hat{a}) > 0$ . Direct computation yields

$$\sigma_l(\hat{a}) = \hat{a}_{nn} \sigma_{l-1}(M) - \sum_{i=1}^{l-1} \hat{a}_{ni} \hat{a}_{in} \sigma_{l-2}(M|i) = \sigma_{l-1}(M) \left[ \hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{in}^2}{\hat{a}_{ii}} \right] > 0,$$

so we have

$$\hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{in}^2}{\hat{a}_{ii}} > 0.$$



This is CASE 1.

For  $k = l$ , we know at  $(x_0, t_0)$

$$\hat{a}_{11} \geq \cdots \geq \hat{a}_{ll} > 0, \quad \hat{a}_{l+1l+1} = \cdots = \hat{a}_{n-1n-1} = 0,$$

and due to  $\hat{a}(x_0, t_0) \geq 0$ , we get

$$\hat{a}_{l+1n} = \cdots = \hat{a}_{n-1n} = 0.$$

Since  $\text{Rank}\{\hat{a}\} = l$ , then  $\sigma_{l+1}(\hat{a}) = 0$ . Direct computation yields

$$\sigma_{l+1}(\hat{a}) = \hat{a}_{nn}\sigma_l(M) - \sum_{i=1}^l \hat{a}_{ni}\hat{a}_{in}\sigma_{l-1}(M|i) = \sigma_l(M)[\hat{a}_{nn} - \sum_{i=1}^l \frac{\hat{a}_{in}^2}{\hat{a}_{ii}}] = 0,$$

so we have

$$\hat{a}_{nn} - \sum_{i=1}^l \frac{\hat{a}_{in}^2}{\hat{a}_{ii}} = 0.$$

This is CASE 2. □

**2.4. Structural conditions (1.3) and (1.5).** Now we discuss the structural conditions (1.3) and (1.5).

First, we introduce the following set to study the matrix  $B^-$ .

**Definition 2.6.** For each  $\theta \in \mathbb{S}^{n-1}$ , define  $\mathcal{A}_\theta^-$  as follows

$$(2.20) \quad \mathcal{A}_\theta^-(\Upsilon) = \left\{ A \in \mathcal{S}^{n+1} \mid A = \begin{pmatrix} \tilde{A} & \mu\theta^T \\ \mu\theta & 0 \end{pmatrix} \text{ with } \tilde{A} \in \mathcal{S}_n^-(\theta) \cap \Upsilon, \mu > 0 \right\}.$$

Properties of  $\mathcal{A}_\theta^-$ ,  $\mathcal{B}_\theta^-$  and their relationship have been studied in [4]. In particular, if  $\theta = (0, \dots, 0, 1)$ ,

$$(2.21) \quad B = \begin{pmatrix} & & 0 & \times \\ & a^{ij} & \vdots & \vdots \\ & & 0 & \times \\ 0 & \cdots & 0 & 0 & s \\ \times & \cdots & \times & s & \chi \end{pmatrix} \in \mathcal{B}_\theta^-(\Upsilon),$$

then

$$(2.22) \quad A = B^{-1} = \begin{pmatrix} & & \times & 0 \\ & a_{ij} & \vdots & \vdots \\ & & \times & 0 \\ \times & \cdots & \times & \times & \mu \\ 0 & \cdots & 0 & \mu & 0 \end{pmatrix}.$$

where the  $(n-1) \times (n-1)$  matrix  $(a_{ij})$  is negative definite and can be assumed diagonal,  $(a^{ij})$  is the inverse matrix of  $(a_{ij})$ ,  $s = B_{n+1,n} = \frac{1}{\mu} > 0$ . The values at the positions denoted by " $\times$ " which are not important in the calculations.

For any given  $V = ((X_{\alpha\beta}), Y, (Z_i), D) \in \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ , we define a quadratic form

$$\begin{aligned}
Q^*(V, V) = & F^{\alpha\beta, \gamma\eta} X_{\alpha\beta} X_{\gamma\eta} + 2F^{\alpha\beta, \iota\iota} \theta_l X_{\alpha\beta} Y + 2F^{\alpha\beta, x_k} X_{\alpha\beta} Z_k + 2F^{\alpha\beta, t} X_{\alpha\beta} D \\
& + F^{u_k, \iota\iota} \theta_k \theta_l Y^2 + 2F^{u_k, x_l} \theta_k Y Z_l + 2F^{u_k, t} \theta_k Y D + F^{x_k, x_l} Z_k Z_l \\
& + 2F^{x_k, t} Z_k D + F^{t, t} D^2 + 2s F^{u_k} \theta_k Y^2 + 6s F^{\alpha\beta} X_{\alpha\beta} Y - 6s F^{\alpha\beta} A_{\alpha\beta} Y^2 \\
& + 2s \sum_{i \in T} \frac{F^{\alpha\beta}}{A_{ii}} [X_{i\alpha} - 2A_{i\alpha} Y] [X_{i\beta} - 2A_{i\beta} Y],
\end{aligned}
\tag{2.23}$$

where the derivative functions of  $F$  are evaluated at  $(s^{-1}\tilde{A}, s^{-1}\theta, u, x, t)$  and  $T := \{1, 2, \dots, n-1\}$ .

Through direct calculations, we can get

**Lemma 2.7.**  *$F$  satisfies the condition (1.3) if and only if for each  $p \in \mathbb{R}^n$*

$$Q^*(V, V) \leq 0, \quad \forall \quad V = ((X_{\alpha\beta}), Y, (Z_i), D) \in \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, \tag{2.24}$$

where the derivative functions of  $F$  are evaluated at  $(s^{-1}\tilde{A}, s^{-1}\theta, u, x, t)$ , and  $Q^*$  is defined in (2.23).

The proof of Lemma 2.7 is similar to the discussion in [2], and we omit it.

**Remark 2.8.**  *$F$  satisfies the condition (1.5) if and only if for each fixed  $p \in \mathbb{R}^n$ , and for any  $\tilde{V} = ((X_{\alpha\beta}), Y, (Z_i), 0) \in \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$*

$$\begin{aligned}
Q^*(\tilde{V}, \tilde{V}) = & F^{\alpha\beta, \gamma\eta} X_{\alpha\beta} X_{\gamma\eta} + 2F^{\alpha\beta, \iota\iota} \theta_l X_{\alpha\beta} Y + 2F^{\alpha\beta, x_k} X_{\alpha\beta} Z_k + F^{u_k, \iota\iota} \theta_k \theta_l Y^2 \\
& + 2F^{u_k, x_l} \theta_k Y Z_l + F^{x_k, x_l} Z_k Z_l + 2s F^{u_k} \theta_k Y^2 + 6s F^{\alpha\beta} X_{\alpha\beta} Y - 6s F^{\alpha\beta} A_{\alpha\beta} Y^2 \\
& + 2s \sum_{i \in T} \frac{F^{\alpha\beta}}{A_{ii}} [X_{i\alpha} - 2A_{i\alpha} Y] [X_{i\beta} - 2A_{i\beta} Y] \\
\leq & 0,
\end{aligned}
\tag{2.25}$$

where the derivative functions of  $F$  are evaluated at  $(s^{-1}\tilde{A}, s^{-1}\theta, u, x, t)$ , and  $Q^*$  is defined in (2.23). Obviously, the condition (1.6) is weaker than the condition (1.4).

**2.5. An auxiliary lemma.** Similarly to the Lemma 2.5 in Bian-Guan[2], we have

**Lemma 2.9.** *Suppose  $W(x) = (W_{ij}(x))_{N \times N} \geq 0$  for every  $x \in \Omega \subset \mathbb{R}^n$ , and  $W_{ij}(x) \in C^{1,1}(\Omega)$ , then for every  $O \subset \subset \Omega$ , there exists a positive constant  $C$  depending only on the  $\text{dist}\{O, \partial\Omega\}$  and  $\|W\|_{C^{1,1}(\Omega)}$  such that*

$$|\nabla W_{ij}| \leq C(W_{ii} W_{jj})^{\frac{1}{4}}, \tag{2.26}$$

for every  $x \in O$  and  $1 \leq i, j \leq N$ .

**Proof:** The same arguments as in the proof of Lemma 2.5 in [2] carry through with a small modification since  $W$  is a general matrix instead of a Hessian of a convex function.

It's known that for any nonnegative  $C^{1,1}$  function  $h$ ,  $|\nabla h(x)| \leq C h^{\frac{1}{2}}(x)$  for all  $x \in O$ , where  $C$  depends only on  $\|h\|_{C^{1,1}(\Omega)}$  and  $\text{dist}\{O, \partial\Omega\}$  (see [29]).

Since  $W(x) \geq 0$ , so we choose  $h(x) = W_{ii}(x) \geq 0$ . Then we can get from the above argument

$$|\nabla W_{ii}| \leq C_1 (W_{ii})^{\frac{1}{2}} = C_1 (W_{ii} W_{ii})^{\frac{1}{4}},$$

so (2.26) holds for  $i = j$ .

Similarly, for  $i \neq j$ , we choose  $h = \sqrt{W_{ii}W_{jj}} \geq 0$ , then we get

$$(2.27) \quad |\nabla \sqrt{W_{ii}W_{jj}}| \leq C_2(\sqrt{W_{ii}W_{jj}})^{\frac{1}{2}} = C_2(W_{ii}W_{jj})^{\frac{1}{4}}.$$

And for  $h = \sqrt{W_{ii}W_{jj}} - W_{ij}$ , we have

$$(2.28) \quad |\nabla(\sqrt{W_{ii}W_{jj}} - W_{ij})| \leq C_3(\sqrt{W_{ii}W_{jj}} - W_{ij})^{\frac{1}{2}} \leq C_3(W_{ii}W_{jj})^{\frac{1}{4}}.$$

So from (2.27) and (2.28), we get

$$\begin{aligned} |\nabla W_{ij}| &= |\nabla \sqrt{W_{ii}W_{jj}} - \nabla(\sqrt{W_{ii}W_{jj}} - W_{ij})| \\ &\leq |\nabla \sqrt{W_{ii}W_{jj}}| + |\nabla(\sqrt{W_{ii}W_{jj}} - W_{ij})| \\ &\leq (C_2 + C_3)(W_{ii}W_{jj})^{\frac{1}{4}}. \end{aligned}$$

So (2.26) holds for  $i \neq j$ . □

**Remark 2.10.** If  $W(x, t) = (W_{ij}(x, t))_{N \times N} \geq 0$  for every  $(x, t) \in \Omega \times (0, T]$ , and  $W_{ij}(x, t) \in C^{1,1}(\Omega \times (0, T])$ , then for every  $O \times (t_0 - \delta, t_0] \subset \subset \Omega \times (0, T]$  with  $t_0 < T$ , there exists a positive constant  $C$  depending only on the  $\text{dist}(O \times (t_0 - \delta, t_0], \partial(\Omega \times (0, T)))$ ,  $t_0$ ,  $\delta$  and  $\|W\|_{C^{1,1}(\Omega \times (0, T])}$  such that

$$(2.29) \quad |DW_{ij}| \leq C(W_{ii}W_{jj})^{\frac{1}{4}},$$

for every  $(x, t) \in O \times (t_0 - \delta, t_0]$  and  $1 \leq i, j \leq N$ . Notice that  $DW_{ij} = (\nabla_x W_{ij}, \partial_t W_{ij})$ . In fact, if  $t_0 = T$ , it only holds

$$(2.30) \quad |\nabla_x W_{ij}| \leq C(W_{ii}W_{jj})^{\frac{1}{4}}.$$

for every  $(x, t) \in O \times (t_0 - \delta, t_0]$  and  $1 \leq i, j \leq N$ .

### 3. CONSTANT RANK THEOREM OF THE SPATIAL SECOND FUNDAMENTAL FORM

In this section, we consider the spatial level sets  $\Sigma^c = \{x \in \Omega | u(x, t) = c\}$ . Since  $u$  is the spacetime quasiconcave solution to fully nonlinear parabolic equation (1.1),  $u$  is also spatial quasiconcave, that is the spatial level sets  $\Sigma^c$  are all convex for  $t \in (0, T]$ , that is the spatial second fundamental form  $a \geq 0$ . We will establish the constant rank theorem for the spatial second fundamental form  $a$  under the structural condition (1.5) as follows.

**Theorem 3.1.** Suppose  $u \in C^{3,1}(\Omega \times (0, T])$  is a spacetime quasiconcave to fully nonlinear parabolic equation (1.1), and  $F$  satisfies conditions (1.2), (1.4) and (1.5). Then the second fundamental form of spatial level sets  $\Sigma^c = \{x \in \Omega | u(x, t) = c\}$  has the same constant rank in  $\Omega$  for any fixed  $t \in (0, T]$ . Moreover, let  $l(t)$  be the minimal rank of the second fundamental form in  $\Omega$ , then  $l(s) \leq l(t)$  for all  $0 < s \leq t \leq T$ .

From the discussion in Section 2, the structural condition (1.5) is weaker than the structural condition (1.3), then Theorem 1.3 holds directly from Theorem 3.1.

In the following of this section, we will prove Theorem 3.1, and discuss some constant rank properties of the spatial second fundamental form  $a$ . And we will prove the constant rank theorem of the spatial fundamental form of the spacetime quasiconcave solutions to the parabolic equations (1.12)-(1.14).

**3.1. Proof of Theorem 3.1.** Suppose  $a(x, t)$  attains minimal rank  $l$  at some point  $(x_0, t_0) \in \Omega \times (0, T]$ . We may assume  $l \leq n - 2$ , otherwise there is nothing to prove. And we assume  $u \in C^{3,1}(\Omega \times (0, T])$  and  $u_n(x_0, t_0) > 0$ . So there is a neighborhood  $\mathcal{O} \times (t_0 - \delta, t_0]$  of  $(x_0, t_0)$ , such that there are  $l$  "good" eigenvalues of  $(a_{ij})$  which are bounded below by a positive constant, and the other  $n - 1 - l$  "bad" eigenvalues of  $(a_{ij})$  are very small. Denote  $G$  be the index set of these "good" eigenvalues and  $B$  be the index set of "bad" eigenvalues. And for any fixed point  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ , we may express  $(a_{ij})$  in a form of (2.3), by choosing  $e_1, \dots, e_{n-1}, e_n$  such that

$$(3.1) \quad |\nabla u(x, t)| = u_n(x, t) > 0 \quad \text{and} \quad (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t).$$

Without loss of generality we assume  $u_{11} \leq u_{22} \leq \dots \leq u_{n-1, n-1}$ . So, at  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ , from (2.2) - (2.4), we have the matrix  $(a_{ij})_{1 \leq i, j \leq n-1}$  is also diagonal, and  $a_{11} \geq a_{22} \geq \dots \geq a_{n-1, n-1}$ . There is a positive constant  $C > 0$  depending only on  $\|u\|_{C^4}$  and  $\mathcal{O} \times (t_0 - \delta, t_0]$ , such that  $a_{11} \geq a_{22} \geq \dots \geq a_{ll} > C$  for all  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ . For convenience we denote  $G = \{1, \dots, l\}$  and  $B = \{l+1, \dots, n-1\}$  be the "good" and "bad" sets of indices respectively. If there is no confusion, we also denote

$$(3.2) \quad G = \{a_{11}, \dots, a_{ll}\}, \quad B = \{a_{l+1, l+1}, \dots, a_{n-1, n-1}\}.$$

Note that for any  $\delta > 0$ , we may choose  $\mathcal{O} \times (t_0 - \delta, t_0]$  small enough such that  $a_{jj} < \delta$  for all  $j \in B$  and  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ .

For each  $c$ , let  $a = (a_{ij})$  be the symmetric Weingarten tensor of  $\Sigma^c$ . Set

$$(3.3) \quad p(a) = \sigma_{l+1}(a_{ij}), \quad q(a) = \begin{cases} \frac{\sigma_{l+2}(a_{ij})}{\sigma_{l+1}(a_{ij})}, & \text{if } \sigma_{l+1}(a_{ij}) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Since we are dealing with general fully nonlinear equation (1.1), as in the case for the convexity of solutions in [2], there are technical difficulties to deal with  $p(a)$  alone. A key idea in [2] is the introduction of function  $q$  as in (3.3) and explore some crucial concavity properties of  $q$ . We consider function

$$(3.4) \quad \phi(x, t) = p(a) + q(a),$$

where  $p$  and  $q$  as in (3.3). We will prove the differential inequality

$$(3.5) \quad \sum_{\alpha, \beta=1}^n F^{\alpha\beta} \phi_{\alpha\beta}(x, t) - \phi_t \leq C(\phi + |\nabla \phi|), \quad \forall (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0],$$

where  $C$  is a positive constant independent of  $\phi$ . Combining with the conditions

$$\phi \geq 0, \quad \text{in } \mathcal{O} \times (t_0 - \delta, t_0],$$

$$\phi(x_0, t_0) = 0,$$

we can get by the strong maximum principle

$$\phi \equiv 0, \quad \text{in } \mathcal{O} \times (t_0 - \delta, t_0].$$

Hence

$$\sigma_{l+1}(a) \equiv 0, \quad \text{in } \mathcal{O} \times (t_0 - \delta, t_0].$$

By the method of continuity, Theorem 3.1 holds. In the following, we prove the differential inequality (3.5).

To get around  $\sigma_{l+1}(a) = 0$  in  $q(a)$ , for  $\varepsilon > 0$  sufficiently small, we instead consider

$$(3.6) \quad \phi_\varepsilon(a) = \phi(a_\varepsilon),$$

where  $a_\varepsilon = a + \varepsilon I$ . We will also denote  $G_\varepsilon = \{a_{ii} + \varepsilon, i \in G\}$ ,  $B_\varepsilon = \{a_{ii} + \varepsilon, i \in B\}$ .

To simplify the notations, we will drop subindex  $\varepsilon$  with the understanding that all the estimates will be independent of  $\varepsilon$ . In this setting, if we pick  $O \times (t_0 - \delta, t_0]$  small enough, there is  $C > 0$  independent of  $\varepsilon$  such that

$$(3.7) \quad \phi(a(x, t)) \geq C\varepsilon, \quad \sigma_1(B) \geq C\varepsilon, \quad \text{for all } (x, t) \in O \times (t_0 - \delta, t_0].$$

In what following, we denote

$$\mathcal{H}_\phi = \sum_{i,j \in B} |\nabla a_{ij}| + \phi.$$

We will use notion  $h = O(\mathcal{H}_\phi)$  if  $|h(x, t)| \leq C\mathcal{H}_\phi$  for  $(x, t) \in O \times (t_0 - \delta, t_0]$  with positive constant  $C$  under control.

For any fixed point  $(x, t) \in O \times (t_0 - \delta, t_0]$ , we choose a coordinate system as in (3.1) so that  $|\nabla u| = u_n > 0$  and the matrix  $(a_{ij}(x, t))$  is diagonal for  $1 \leq i, j \leq n-1$  and semipositive definite. From the definition of  $\phi$ , we get

$$\phi \geq \sigma_l(G) \sum_{i \in B} a_{ii} \geq 0,$$

so

$$(3.8) \quad a_{ii} = O(\phi) = O(\mathcal{H}_\phi), \quad \forall i \in B.$$

And from (2.2) - (2.4), we get

$$a_{ii} = -\frac{h_{ii}}{u_n^3} = -\frac{u_{ii}}{u_n},$$

so

$$(3.9) \quad h_{ii} = O(\mathcal{H}_\phi), u_{ii} = O(\mathcal{H}_\phi), \quad \forall i \in B.$$

From the definition of  $a_{ij}$ , and  $u_k = 0$  for  $k = 1, \dots, n-1$ , we can get

$$\begin{aligned} a_{ij,\alpha} &= \left(-\frac{|u_n|}{|\nabla u|u_n^3}\right)_\alpha h_{ij} + \left(-\frac{|u_n|}{|\nabla u|u_n^3}\right) h_{ij,\alpha} \\ &= 3u_n^{-4} u_n^2 u_{ij} - u_n^{-3} [u_n^2 u_{ij\alpha} + 2u_n u_{n\alpha} u_{ij} - u_{i\alpha} u_n u_{jn} - u_{j\alpha} u_n u_{in}] \\ (3.10) \quad &= -u_n^{-2} [u_n u_{ij\alpha} - u_{n\alpha} u_{ij} - u_{i\alpha} u_{jn} - u_{j\alpha} u_{in}], \end{aligned}$$

so for  $i, j \in B$ , we get

$$(3.11) \quad u_{ij\alpha} = O(\mathcal{H}_\phi), \quad \forall \alpha < n,$$

$$(3.12) \quad u_n u_{ijn} = 2u_{in} u_{jn} + O(\mathcal{H}_\phi).$$

In fact, from (2.6) - (2.8),

$$\hat{a}_{jj} = -\frac{|u_t|}{|Du|u_t^3} \hat{h}_{jj} = -\frac{u_{jj}}{|Du|} = O(\mathcal{H}_\phi), \quad \forall j \in B,$$

and from the spacetime convexity, we can get

$$\hat{a}_{jn}^2 = \left[ -\frac{|u_t|}{|Du|u_t^3} \frac{1}{\hat{W}} \hat{h}_{jn} \right]^2 \leq \hat{a}_{jj} \hat{a}_{nn} = O(\mathcal{H}_\phi), \quad \forall j \in B,$$

so it yields

$$(3.13) \quad \hat{h}_{jn}^2 = O(\mathcal{H}_\phi), \quad \forall j \in B.$$

Following the proof of Lemma 2.1 in [12], we can get

$$(3.14) \quad \begin{aligned} \phi_t &= \sum_{ij=1}^{n-1} \frac{\partial \phi}{\partial a_{ij}} a_{ij,t} \\ &= -u_n^{-3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] [u_n^2 u_{jjt} - 2u_n u_{jn} u_{jt}] + O(\mathcal{H}_\phi) \end{aligned}$$

and

$$(3.15) \quad \begin{aligned} \sum_{\alpha, \beta=1}^n F^{\alpha\beta} \phi_{\alpha\beta} &= \sum_{\alpha, \beta=1}^n F^{\alpha\beta} \left[ \sum_{ij=1}^{n-1} \frac{\partial \phi}{\partial a_{ij}} a_{ij, \alpha\beta} + \sum_{ijkl=1}^{n-1} \frac{\partial^2 \phi}{\partial a_{ij} \partial a_{kl}} a_{ij, \alpha} a_{kl, \beta} \right] \\ &= u_n^{-3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] \left[ -u_n^2 \sum_{\alpha, \beta=1}^n F^{\alpha\beta} u_{\alpha\beta jj} + 6u_n u_{nj} \sum_{\alpha, \beta=1}^n F^{\alpha\beta} u_{\alpha\beta j} \right. \\ &\quad \left. - 6u_n^2 \sum_{\alpha, \beta=1}^n F^{\alpha\beta} u_{\alpha\beta} \right] \\ &\quad + 2u_n^{-3} \sum_{j \in B, i \in G} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] \sum_{\alpha, \beta=1}^n F^{\alpha\beta} \frac{1}{u_{ii}} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}] [u_n u_{ij\beta} - 2u_{i\beta} u_{jn}] \\ &\quad - \frac{1}{\sigma_1^3(B)} \sum_{\alpha, \beta=1}^n \sum_{i \in B} F^{\alpha\beta} [\sigma_1(B) a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \alpha}] [\sigma_1(B) a_{ii, \beta} - a_{ii} \sum_{j \in B} a_{jj, \beta}] \\ &\quad - \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta=1}^n \sum_{i \neq j \in B} F^{\alpha\beta} a_{ij, \alpha} a_{ij, \beta} + O(\mathcal{H}_\phi). \end{aligned}$$

Hence

$$\begin{aligned}
& \sum_{\alpha, \beta=1}^n F^{\alpha\beta} \phi_{\alpha\beta} - \phi_t \\
= & -u_n^{-3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] \left[ u_n^2 \left( \sum_{\alpha, \beta=1}^n F^{\alpha\beta} u_{j\alpha\beta} - u_{j\beta} \right) + 2u_n u_{jn} u_{jt} \right. \\
& \quad \left. - 6u_n u_{jn} \sum_{\alpha, \beta=1}^n F^{\alpha\beta} u_{j\alpha\beta} + 6u_{jn}^2 \sum_{\alpha, \beta=1}^n F^{\alpha\beta} u_{\alpha\beta} \right] \\
& + 2u_n^{-3} \sum_{j \in B, i \in G} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] \sum_{\alpha, \beta=1}^n F^{\alpha\beta} \frac{1}{u_{ii}} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}] [u_n u_{ij\beta} - 2u_{i\beta} u_{jn}] \\
(3.16) \quad & - \frac{1}{\sigma_1^3(B)} \sum_{\alpha, \beta=1}^n \sum_{i \in B} F^{\alpha\beta} [\sigma_1(B) a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \alpha}] [\sigma_1(B) a_{ii, \beta} - a_{ii} \sum_{j \in B} a_{jj, \beta}] \\
& - \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta=1}^n \sum_{i \neq j, i, j \in B} F^{\alpha\beta} a_{ij, \alpha} a_{ij, \beta} + O(\mathcal{H}_\phi).
\end{aligned}$$

From  $\hat{h}_{jn} = u_t^2 u_{jn} - u_n u_t u_{jt}$ , we have

$$\begin{aligned}
2u_n u_{jn} u_{jt} &= -\frac{1}{u_t} \left[ \left( \frac{\hat{h}_{jn}}{u_t} \right)^2 - (u_t u_{jn})^2 - (u_n u_{jt})^2 \right] \\
(3.17) \quad &= O(\mathcal{H}_\phi) + \frac{1}{u_t} [(u_t u_{jn})^2 + (u_n u_{jt})^2],
\end{aligned}$$

where the second "=" holds from (3.13).

For each  $j \in B$ , differentiating equation (1.2) in  $e_j$  direction at  $x$ ,

$$\begin{aligned}
u_{j\beta} &= \sum_{kl=1}^n F^{kl} u_{klj\beta} + \sum_{i=1}^n F^{u_i} u_{j\beta i} + F^u u_{j\beta} \\
&+ \sum_{klpq=1}^n F^{kl, pq} u_{klj} u_{pqj} + 2 \sum_{ikl=1}^n F^{kl, u_i} u_{klj} u_{ij} + 2 \sum_{kl=1}^n F^{kl, u} u_{klj} u_j \\
&+ 2 \sum_{kl=1}^n F^{kl, x_j} u_{klj} + \sum_{ik=1}^n F^{u_i, u_k} u_{ij} u_{kj} + 2 \sum_{i=1}^n F^{u_i, u} u_{ij} u_j + 2 \sum_{i=1}^n F^{u_i, x_j} u_{ij} \\
&+ F^{u, u} u_j^2 + 2F^{u, x_j} u_j + F^{x_j, x_j} \\
&= \sum_{kl=1}^n F^{kl} u_{klj\beta} + 2 \frac{F^{u_n}}{u_n} u_{jn}^2 + \sum_{klpq=1}^n F^{kl, pq} u_{klj} u_{pqj} + 2 \sum_{kl=1}^n F^{kl, u_n} u_{klj} u_{jn} \\
(3.18) \quad &+ 2 \sum_{kl=1}^n F^{kl, x_j} u_{klj} + F^{u_n, u_n} u_{jn}^2 + 2F^{u_n, x_j} u_{jn} + F^{x_j, x_j} + O(\mathcal{H}_\phi).
\end{aligned}$$

Set

$$\begin{aligned}
 (3.19) \quad Q_j &= \sum_{klpq=1}^n F^{kl,pq} u_{klj} u_{pqj} u_n^2 + 2 \sum_{kl=1}^n F^{kl,u_n} u_{klj} u_{jn} u_n^2 + 2 \sum_{kl=1}^n F^{kl,x_j} u_{klj} u_n^2 + F^{u_n,u_n} u_{jn}^2 u_n^2 \\
 &+ 2 F^{u_n,x_j} u_{jn} u_n^2 + F^{x_j,x_j} u_n^2 + 2 F^{u_n} u_n u_{jn}^2 + 6 u_n u_{jn} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{j\alpha\beta} - 6 u_{jn}^2 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta} \\
 &+ 2 \sum_{i \in G} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} \frac{1}{u_{ii}} [u_n u_{ij\alpha} - 2 u_{i\alpha} u_{jn}] [u_n u_{ij\beta} - 2 u_{i\beta} u_{jn}],
 \end{aligned}$$

and denote

$$\begin{aligned}
 s &= \frac{1}{u_n} = \frac{1}{|\nabla u|}, A_{ij} = s u_{ij} = \frac{u_{ij}}{u_n}, \theta = (0, 0, \dots, 0, 1); \\
 X_{\alpha\beta} &= 2 u_{\alpha\beta} u_{jn}, \quad \alpha \in B \text{ or } \beta \in B; \\
 X_{\alpha\beta} &= u_{\alpha\beta j} u_n, \quad \text{otherwise}; \\
 Y &= u_{jn} u_n; \\
 Z_i &= \delta_{ij}, \quad i = 1, 2, \dots, n; \\
 \tilde{V} &= ((X_{\alpha\beta}), Y, (Z_i), 0) \in \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R};
 \end{aligned}$$

then we can get

$$X_{i\alpha} - 2 A_{i\alpha} Y = 0, \quad i \in B.$$

So it yields

$$(3.20) \quad Q_j = Q^*(\tilde{V}, \tilde{V}),$$

where  $Q^*(\tilde{V}, \tilde{V})$  is defined in (2.25).

From (3.16) - (3.18)), it yields

$$\begin{aligned}
 (3.21) \quad & F^{\alpha\beta} \phi_{\alpha\beta} - \phi_t \\
 &= u_n^{-3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] (Q_j - 2 u_n u_{jn} u_{ji}) \\
 &\quad - \frac{1}{\sigma_1^3(B)} \sum_{\alpha,\beta=1}^n \sum_{i \in B} F^{\alpha\beta} [\sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\alpha}] [\sigma_1(B) a_{ii,\beta} - a_{ii} \sum_{j \in B} a_{jj,\beta}] \\
 &\quad - \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^n \sum_{i \neq j, i, j \in B} F^{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} + O(\mathcal{H}_\phi) \\
 &\leq u_n^{-3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] Q_j \\
 &\quad - \frac{1}{\sigma_1^3(B)} \sum_{\alpha,\beta=1}^n \sum_{i \in B} F^{\alpha\beta} [\sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\alpha}] [\sigma_1(B) a_{ii,\beta} - a_{ii} \sum_{j \in B} a_{jj,\beta}] \\
 &\quad - \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^n \sum_{i \neq j, i, j \in B} F^{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} + O(\mathcal{H}_\phi).
 \end{aligned}$$



From the structural condition (1.6) (i.e. Remark 2.8), it implies

$$\mathcal{Q}^*(\tilde{V}, \tilde{V}) \leq 0.$$

so for  $j \in B$ , we get

$$(3.22) \quad \mathcal{Q}_j = \mathcal{Q}^*(\tilde{V}, \tilde{V}) \leq 0.$$

Condition (1.2) implies

$$(3.23) \quad (F^{\alpha\beta}) \geq \delta_0 I_n, \quad \text{for some } \delta_0 > 0, \text{ and } \forall x \in \mathcal{O}.$$

Set

$$V_{i\alpha} = \sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\alpha}.$$

Combining (3.21), (3.22) and (3.23),

$$(3.24) \quad F^{\alpha\beta} \phi_{\alpha\beta} \leq C(\phi + \sum_{i,j \in B} |\nabla a_{ij}|) - \delta_0 \left[ \frac{\sum_{i \neq j \in B, \alpha=1}^n a_{ij,\alpha}^2}{\sigma_1(B)} + \frac{\sum_{i \in B, \alpha=1}^n V_{i\alpha}^2}{\sigma_1^3(B)} \right].$$

By Lemma 3.3 in [2], for each  $M \geq 1$ , for any  $M \geq |\gamma_i| \geq \frac{1}{M}$ , there is a constant  $C$  depending only on  $n$  and  $M$  such that,  $\forall \alpha$ ,

$$(3.25) \quad \sum_{i,j \in B} |a_{ij,\alpha}| \leq C(1 + \frac{1}{\delta_0^2})(\sigma_1(B) + |\sum_{i \in B} \gamma_i a_{ii,\alpha}|) + \frac{\delta_0}{2} \left[ \frac{\sum_{i \neq j \in B} |a_{ij,\alpha}|^2}{\sigma_1(B)} + \frac{\sum_{i \in B} V_{i\alpha}^2}{\sigma_1^3(B)} \right].$$

Taking  $\gamma_i = \sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}$  for each  $i \in B$ , the Newton-MacLaurine inequality implies

$$\sigma_l(G) + 1 \geq \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \geq \sigma_l(G), \quad \forall j \in B.$$

and

$$(3.26) \quad \phi_\alpha = \sum_{ij=1}^{n-1} \frac{\partial \phi}{\partial a_{ij}} a_{ij,\alpha} = \sum_{i \in B} \gamma_i a_{ii,\alpha} + O(\phi).$$

Therefore we conclude from (3.25) and (3.26) that  $\sum_{i,j \in B} |\nabla a_{ij}|$  can be controlled by the rest terms on the right hand side in (3.24) and  $\phi + |\nabla \phi|$ . So (3.5) holds, and the proof of Theorem 3.1 is complete.  $\square$

**3.2. Constant rank properties of  $a$ .** In the proof of Theorem 3.1, we can get for any  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$  with the suitable coordinate (3.1),

$$\begin{aligned}
 & \sum_{\alpha, \beta=1}^n F^{\alpha\beta} \phi_{\alpha\beta} - \phi_t \\
 = & u_n^{-3} \sum_{j \in B, i \in G} \left[ \sigma_1(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] \left( Q_j - \frac{1}{u_t} [(u_t u_{jn})^2 + (u_n u_{jt})^2] \right) \\
 (3.27) \quad & - \frac{1}{\sigma_1^3(B)} \sum_{\alpha, \beta=1}^n \sum_{i \in B} F^{\alpha\beta} [\sigma_1(B) a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \alpha}] [\sigma_1(B) a_{ii, \beta} - a_{ii} \sum_{j \in B} a_{jj, \beta}] \\
 & - \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta=1}^n \sum_{i \neq j, i, j \in B} F^{\alpha\beta} a_{ij, \alpha} a_{ij, \beta} + O(\mathcal{H}_\phi) \\
 \leq & C(\phi + |\nabla \phi|),
 \end{aligned}$$

and by the strong maximum principle,

$$(3.28) \quad \phi = 0 \quad \text{for } (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].$$

So it must have for any  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$  with the suitable coordinate (3.1)

$$(3.29) \quad a_{jj} = 0, \quad \text{for } j \in B.$$

In fact, we can get more information from the differential inequality, and the constant rank properties is as follows

**Corollary 3.2.** *For any  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0)$  with the suitable coordinate (3.1)*

$$(3.30) \quad u_{jn} = u_{jt} = 0, |Du_j| = 0, \quad \text{for } j \in B;$$

$$(3.31) \quad \sum_{kl=1}^n F^{kl} u_{kljj} - u_{jjt} = 2 \sum_{i \in G} \sum_{\alpha, \beta=1}^n F^{\alpha\beta} \frac{u_n^2 u_{ij\alpha} u_{ij\beta}}{u_{ii}}, \quad \text{for } j \in B;$$

$$(3.32) \quad |Du_{ij}| = 0, \quad \text{for } i \in B, j = 1, 2, \dots, n-1.$$

PROOF. For  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0)$  with the suitable coordinate (3.1), we have from (2.3) and (3.29)

$$u_{jj} = 0, \quad \text{for } j \in B.$$

and from (3.28) and (3.27), we get for  $j \in B$ ,

$$Q_j = 0,$$

$$u_t u_{jn} = u_n u_{jt} = 0.$$

So

$$(3.33) \quad u_{jn} = u_{jt} = 0, \quad \text{for } j \in B,$$

then

$$(3.34) \quad |Du_j| = 0, \quad \text{for } j \in B.$$

From the definition of  $Q_j$ , and (3.28), (3.32), (3.33), we get

$$\begin{aligned}
0 = Q_j &= \sum_{klmn=1}^n F^{kl,mn} u_{klj} u_{mnj} u_n^2 + 2 \sum_{kl=1}^n F^{kl,u_n} u_{klj} u_{jn} u_n^2 + 2 \sum_{kl=1}^n F^{kl,x_j} u_{klj} u_n^2 \\
&\quad + F^{u_n,u_n} u_{jn}^2 u_n^2 + 2 F^{u_n,x_j} u_{jn} u_n^2 + F^{x_j,x_j} u_n^2 + 2 \sum_{i \in G} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} \frac{u_n^2 u_{ij\alpha} u_{ij\beta}}{u_{ii}} \\
&= u_{jjt} - \sum_{kl=1}^n F^{kl} u_{kljj} + 2 \sum_{i \in G} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} \frac{u_n^2 u_{ij\alpha} u_{ij\beta}}{u_{ii}}, \quad \text{for } j \in B.
\end{aligned}$$

Also, we can get from (3.28), and Lemma 2.9 (i.e. Remark 2.10)

$$|Da_{ij}| = 0, \quad \text{for } i \in B, j = 1, 2, \dots, n-1,$$

then from (3.10) and (3.32)

$$|Du_{ij}| = 0, \quad \text{for } i \in B, j = 1, 2, \dots, n-1.$$

So the proof is complete.  $\square$

**3.3. Constant rank theorem of the spatial fundamental form for the equation (1.12).** In this subsection, we consider the  $p$ -Laplacian parabolic equation, that is

$$(3.35) \quad u_t = \operatorname{div}(|\nabla u|^{p-2} \nabla u) = L_{kl}(\nabla u) u_{kl}, \text{ in } \Omega \times (0, T],$$

where

$$(3.36) \quad L_{kl}(\nabla u) = |\nabla u|^{p-2} \delta_{kl} + (p-2) |\nabla u|^{p-4} u_k u_l.$$

It is easy to know the equation (3.35) is parabolic when  $p > 1$  and  $|\nabla u| > 0$  in  $\Omega \times [0, T]$ . We will establish the constant rank theorem for the spatial second fundamental form  $a$  as follows.

**Theorem 3.3.** *Suppose  $u \in C^{3,1}(\Omega \times (0, T])$  is a spacetime quasiconcave to the parabolic equation (3.35) and satisfies (1.4). Then the second fundamental form of spatial level sets  $\Sigma^c = \{x \in \Omega | u(x, t) = c\}$  has the same constant rank in  $\Omega$  for any fixed  $t \in (0, T]$ . Moreover, let  $l(t)$  be the minimal rank of the second fundamental form in  $\Omega$ , then  $l(s) \leq l(t)$  for all  $0 < s \leq t \leq T$ .*

**PROOF.** The proof is similar to the the proof of Theorem 3.1, with some modifications.

Suppose  $a(x, t)$  attains minimal rank  $l$  at some point  $(x_0, t_0) \in \Omega \times (0, T]$ . We may assume  $l \leq n-2$ , otherwise there is nothing to prove. So there is a small neighborhood  $O \times (t_0 - \delta, t_0]$  of  $(x_0, t_0)$ , such that there are  $l$  "good" eigenvalues of  $(a_{ij})$  which are bounded below by a positive constant, and the other  $n-1-l$  "bad" eigenvalues of  $(a_{ij})$  are very small. Denote  $G$  be the index set of these "good" eigenvalues and  $B$  be the index set of "bad" eigenvalues. We will prove the differential inequality

$$(3.37) \quad \sum_{\alpha,\beta=1}^n L_{\alpha\beta} \phi_{\alpha\beta}(x, t) - \phi_t \leq C(\phi + |\nabla \phi|), \quad \forall (x, t) \in O \times (t_0 - \delta, t_0],$$

where  $\phi$  is defined in (3.4) and  $C$  is a positive constant independent of  $\phi$ . Then by the strong maximum principle and the method of continuity, Theorem 3.3 holds.

For any fixed point  $(x, t) \in O \times (t_0 - \delta, t_0]$ , we may express  $(a_{ij})$  in a form of (2.3), by choosing  $e_1, \dots, e_{n-1}, e_n$  such that

$$(3.38) \quad |\nabla u(x, t)| = u_n(x, t) > 0 \text{ and } (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t).$$

Following the proof of Theorem 3.1, we get from (3.21)

$$(3.39) \quad \begin{aligned} & L_{\alpha\beta}\phi_{\alpha\beta} - \phi_t \\ = & u_n^{-3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] (Q_j - 2u_n u_{jn} u_{jt}) \\ & - \frac{1}{\sigma_1^3(B)} \sum_{\alpha, \beta=1}^n \sum_{i \in B} L_{\alpha\beta} [\sigma_1(B) a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \alpha}] [\sigma_1(B) a_{ii, \beta} - a_{ii} \sum_{j \in B} a_{jj, \beta}] \\ & - \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta=1}^n \sum_{i \neq j, i, j \in B} L_{\alpha\beta} a_{ij, \alpha} a_{ij, \beta} + O(\mathcal{H}_\phi), \end{aligned}$$

where

$$\begin{aligned} Q_j = & 2 \sum_{kl=1}^n \frac{\partial L_{kl}}{\partial u_n} u_{klj} u_{jn} u_n^2 + \sum_{kl=1}^n \frac{\partial^2 L_{kl}}{\partial u_n^2} u_{kl} u_{jn}^2 u_n^2 \\ & + 2 \sum_{kl=1}^n \frac{\partial L_{kl}}{\partial u_n} u_{kl} u_n u_{jn}^2 + 6u_n u_{jn} \sum_{kl=1}^n L_{kl} u_{jkl} - 6u_{jn}^2 \sum_{kl=1}^n L_{kl} u_{kl} \\ & + 2 \sum_{i \in G} \sum_{\alpha, \beta=1}^n \frac{1}{u_{ii}} L_{\alpha\beta} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}] [u_n u_{ij\beta} - 2u_{i\beta} u_{jn}]. \end{aligned}$$

Under the coordinate (3.38), we get

$$(3.40) \quad L_{kl} = 0, k \neq l; \quad L_{kk} = u_n^{p-2}, k < n; \quad L_{nn} = (p-1)u_n^{p-2};$$

$$(3.41) \quad \frac{\partial L_{kl}}{\partial u_n} = 0, \quad k \neq l;$$

$$(3.42) \quad \frac{\partial L_{kk}}{\partial u_n} = (p-2)u_n^{p-3} = (p-2)\frac{L_{kk}}{u_n}, \quad k < n;$$

$$(3.43) \quad \frac{\partial L_{nn}}{\partial u_n} = (p-1)(p-2)u_n^{p-3} = (p-2)\frac{L_{nn}}{u_n}.$$

and

$$(3.44) \quad \frac{\partial^2 L_{kl}}{\partial u_n^2} = 0, k \neq l;$$

$$(3.45) \quad \frac{\partial^2 L_{kk}}{\partial u_n^2} = (p-2)(p-3)u_n^{p-4} = (p-2)(p-3)\frac{L_{kk}}{u_n^2}, k < n;$$

$$(3.46) \quad \frac{\partial^2 L_{nn}}{\partial u_n^2} = (p-1)(p-2)(p-3)u_n^{p-4} = (p-2)(p-3)\frac{L_{nn}}{u_n^2}.$$

From the equation (3.35), we know

$$u_t = L_{kk} u_{kk},$$

and for  $j \in B$

$$\begin{aligned}
 u_{tj} &= L_{kk}u_{kkj} + \frac{\partial L_{kl}}{\partial u_p}u_{pj}u_{kl} = L_{kk}u_{kkj} + \frac{\partial L_{kl}}{\partial u_j}u_{jj}u_{kl} + \frac{\partial L_{kl}}{\partial u_n}u_{nj}u_{kl} \\
 &= L_{kk}u_{kkj} + O(\mathcal{H}_\phi) + \frac{\partial L_{kk}}{\partial u_n}u_{nj}u_{kk} = L_{kk}u_{kkj} + (p-2)\frac{L_{kk}}{u_n}u_{nj}u_{kk} + O(\mathcal{H}_\phi) \\
 (3.47) \quad &= L_{kk}u_{kkj} + (p-2)\frac{u_t}{u_n}u_{nj} + O(\mathcal{H}_\phi).
 \end{aligned}$$

And from (3.13), we get

$$(3.48) \quad \hat{h}_{jn}^2 = O(\mathcal{H}_\phi), \forall j \in B.$$

So

$$\begin{aligned}
 Q_j &= 2 \sum_{k=1}^n (p-2) \frac{L_{kk}}{u_n} u_{kkj} u_{jn} u_n^2 + \sum_{k=1}^n (p-2)(p-3) \frac{L_{kk}}{u_n^2} u_{kk} u_{jn}^2 u_n^2 \\
 &\quad + 2 \sum_{k=1}^n (p-2) \frac{L_{kk}}{u_n} u_{kk} u_n u_{jn}^2 + 6u_n u_{jn} \sum_{k=1}^n L_{kk} u_{kkj} - 6u_{jn}^2 u_t \\
 &\quad + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}]^2 \\
 &= 2(p-2)[u_{tj} - (p-2)\frac{u_t}{u_n}u_{nj}]u_{jn}u_n + (p-2)(p-3)u_t u_{jn}^2 \\
 &\quad + 2(p-2)u_t u_{jn}^2 + 6u_n u_{jn} [u_{tj} - (p-2)\frac{u_t}{u_n}u_{nj}] - 6u_{jn}^2 u_t \\
 &\quad + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}]^2 + O(\mathcal{H}_\phi) \\
 &= (2p+2)u_n u_{jn} u_{tj} - (p^2+p)u_t u_{jn}^2 \\
 &\quad + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}]^2 + O(\mathcal{H}_\phi).
 \end{aligned}$$

Hence

$$\begin{aligned}
Q_j - 2u_n u_{jn} u_{tj} &= 2p u_n u_{jn} u_{tj} - (p^2 + p) u_t u_{jn}^2 \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}]^2 + O(\mathcal{H}_\phi) \\
&= 2p u_{jn} [u_t u_{jn} - \frac{\hat{h}_{jn}}{u_t}] - (p^2 + p) u_t u_{jn}^2 \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}]^2 + O(\mathcal{H}_\phi) \\
&= \frac{p}{u_t} \left[ \frac{1}{p-1} \frac{\hat{h}_{jn}^2}{u_t^2} - (p-1) (u_t u_{jn} + \frac{1}{p-1} \frac{\hat{h}_{jn}}{u_t})^2 \right] \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}]^2 + O(\mathcal{H}_\phi) \\
&\leq \frac{p}{u_t} \cdot \frac{1}{p-1} \frac{\hat{h}_{jn}^2}{u_t^2} + O(\mathcal{H}_\phi) = O(\mathcal{H}_\phi).
\end{aligned}$$

So we can get

$$\begin{aligned}
L_{\alpha\beta} \phi_{\alpha\beta} - \phi_t &\leq C(\phi + \sum_{i,j \in B} |\nabla a_{ij}|) \\
&\quad - \frac{1}{\sigma_1^3(B)} \sum_{\alpha,\beta=1}^n \sum_{i \in B} L_{\alpha\beta} [\sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\alpha}] [\sigma_1(B) a_{ii,\beta} - a_{ii} \sum_{j \in B} a_{jj,\beta}] \\
&\quad - \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^n \sum_{i \neq j, i,j \in B} L_{\alpha\beta} a_{ij,\alpha} a_{ij,\beta},
\end{aligned} \tag{3.49}$$

Following the proof of Theorem 3.1, we get (3.37). □

**Remark 3.4.** The constant rank properties (that is, Corollary 3.2) still holds for the equation (1.12).

**3.4. Constant rank theorem of the spatial fundamental form for the equation (1.13).** In this subsection, we consider the mean curvature parabolic equation, that is

$$(3.50) \quad u_t = \operatorname{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = m_{kl}(\nabla u) u_{kl}, \text{ in } \Omega \times (0, T],$$

where

$$(3.51) \quad m_{kl}(\nabla u) = (1 + |\nabla u|^2)^{-\frac{1}{2}} \delta_{kl} - (1 + |\nabla u|^2)^{-\frac{3}{2}} u_k u_l.$$

We will establish the constant rank theorem for the spatial second fundamental form  $a$  as follows.

**Theorem 3.5.** Suppose  $u \in C^{3,1}(\Omega \times (0, T])$  is a spacetime quasiconcave to the parabolic equation (3.50) and satisfies (1.4). Then the second fundamental form of spatial level sets  $\Sigma^c = \{x \in \Omega | u(x, t) = c\}$  has the same constant rank in  $\Omega$  for any fixed  $t \in (0, T]$ . Moreover, let  $l(t)$  be the minimal rank of the second fundamental form in  $\Omega$ , then  $l(s) \leq l(t)$  for all  $0 < s \leq t \leq T$ .

PROOF. The proof is similar to the the proof of Theorem 3.1 and Theorem 3.3, with some modifications.

Suppose  $a(x, t)$  attains minimal rank  $l$  at some point  $(x_0, t_0) \in \Omega \times (0, T]$ . We may assume  $l \leq n - 2$ , otherwise there is nothing to prove. So there is a small neighborhood  $O \times (t_0 - \delta, t_0]$  of  $(x_0, t_0)$ , such that there are  $l$  "good" eigenvalues of  $(a_{ij})$  which are bounded below by a positive constant, and the other  $n - 1 - l$  "bad" eigenvalues of  $(a_{ij})$  are very small. Denote  $G$  be the index set of these "good" eigenvalues and  $B$  be the index set of "bad" eigenvalues. We will prove the differential inequality

$$(3.52) \quad \sum_{\alpha, \beta=1}^n m_{\alpha\beta} \phi_{\alpha\beta}(x, t) - \phi_t \leq C(\phi + |\nabla \phi|), \quad \forall (x, t) \in O \times (t_0 - \delta, t_0],$$

where  $\phi$  is defined in (3.4) and  $C$  is a positive constant independent of  $\phi$ . Then by the strong maximum principle and the method of continuity, Theorem 3.5 holds.

For any fixed point  $(x, t) \in O \times (t_0 - \delta, t_0]$ , we may express  $(a_{ij})$  in a form of (2.3), by choosing  $e_1, \dots, e_{n-1}, e_n$  such that

$$(3.53) \quad |\nabla u(x, t)| = u_n(x, t) > 0 \text{ and } (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t).$$

Following the proof of Theorem 3.1, we get from (3.21)

$$(3.54) \quad \begin{aligned} & m_{\alpha\beta} \phi_{\alpha\beta} - \phi_t \\ = & u_n^{-3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] (Q_j - 2u_n u_{jn} u_{jt}) \\ & - \frac{1}{\sigma_1^3(B)} \sum_{\alpha, \beta=1}^n \sum_{i \in B} m_{\alpha\beta} [\sigma_1(B) a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \alpha}] [\sigma_1(B) a_{ii, \beta} - a_{ii} \sum_{j \in B} a_{jj, \beta}] \\ & - \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta=1}^n \sum_{i \neq j, i, j \in B} m_{\alpha\beta} a_{ij, \alpha} a_{ij, \beta} + O(\mathcal{H}_\phi), \end{aligned}$$

where

$$\begin{aligned} Q_j = & 2 \sum_{kl=1}^n \frac{\partial m_{kl}}{\partial u_n} u_{klj} u_{jn} u_n^2 + \sum_{kl=1}^n \frac{\partial^2 m_{kl}}{\partial u_n^2} u_{kl} u_{jn}^2 u_n^2 \\ & + 2 \sum_{kl=1}^n \frac{\partial m_{kl}}{\partial u_n} u_{kl} u_n u_{jn}^2 + 6u_n u_{jn} \sum_{kl=1}^n m_{kl} u_{jkl} - 6u_{jn}^2 \sum_{kl=1}^n m_{kl} u_{kl} \\ & + 2 \sum_{i \in G} \sum_{\alpha, \beta=1}^n \frac{1}{u_{ii}} m_{\alpha\beta} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}] [u_n u_{ij\beta} - 2u_{i\beta} u_{jn}]. \end{aligned}$$

Under the coordinate (3.53), we get

$$m_{kl} = 0, \quad k \neq l; \quad m_{kk} = (1 + u_n^2)^{-\frac{1}{2}}, \quad k < n; \quad m_{nn} = (1 + u_n^2)^{-\frac{3}{2}};$$

$$\begin{aligned}\frac{\partial m_{kl}}{\partial u_n} &= 0, \quad k \neq l; \\ \frac{\partial m_{kk}}{\partial u_n} &= -(1 + u_n^2)^{-\frac{3}{2}} u_n, \quad k < n; \\ \frac{\partial m_{nn}}{\partial u_n} &= -3(1 + u_n^2)^{-\frac{5}{2}} u_n;\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 m_{kl}}{\partial u_n^2} &= 0, \quad k \neq l; \\ \frac{\partial^2 m_{kk}}{\partial u_n^2} &= -(1 + u_n^2)^{-\frac{3}{2}} + 3(1 + u_n^2)^{-\frac{5}{2}} u_n^2, \quad k < n; \\ \frac{\partial^2 m_{nn}}{\partial u_n^2} &= -3(1 + u_n^2)^{-\frac{5}{2}} + 15(1 + u_n^2)^{-\frac{7}{2}} u_n^2.\end{aligned}$$

From (3.9), and (3.11)

$$(3.55) \quad u_{kk} = O(\mathcal{H}_\phi), \quad u_{kkj} = O(\mathcal{H}_\phi), \quad \forall k \in B, j \in B,$$

From (3.13), we get

$$(3.56) \quad \hat{h}_{jn}^2 = O(\mathcal{H}_\phi), \quad \forall j \in B.$$

From the equation (3.50), we know

$$u_t = \sum_{k=1}^n m_{kk} u_{kk} = (1 + u_n^2)^{-\frac{1}{2}} \sum_{k=1}^{n-1} u_{kk} + (1 + u_n^2)^{-\frac{3}{2}} u_{nn},$$

so we get

$$\begin{aligned}u_t - (1 + u_n^2)^{-\frac{3}{2}} u_{nn} &= (1 + u_n^2)^{-\frac{1}{2}} \sum_{k=1}^{n-1} u_{kk} \\ &= (1 + u_n^2)^{-\frac{1}{2}} \sum_{k \in G} u_{kk} + O(\mathcal{H}_\phi),\end{aligned}$$

and since  $u_{kk} \leq 0$  for  $k < n$ , it yields

$$(1 + u_n^2)^{-\frac{3}{2}} u_{nn} \geq u_t.$$

Hence we can get

$$\begin{aligned}\sum_{kl=1}^n \frac{\partial m_{kl}}{\partial u_n} u_{kl} &= \sum_{k=1}^n \frac{\partial m_{kk}}{\partial u_n} u_{kk} = \sum_{k=1}^{n-1} \frac{\partial m_{kk}}{\partial u_n} u_{kk} + \frac{\partial m_{nn}}{\partial u_n} u_{nn} \\ &= -(1 + u_n^2)^{-\frac{3}{2}} u_n \sum_{k=1}^{n-1} u_{kk} - 3(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nn} \\ &= -(1 + u_n^2)^{-1} u_n [u_t - (1 + u_n^2)^{-\frac{3}{2}} u_{nn}] - 3(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nn} \\ &= -(1 + u_n^2)^{-1} u_t u_n - 2(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nn}.\end{aligned}$$



and

$$\begin{aligned}
\sum_{kl=1}^n \frac{\partial^2 m_{kl}}{\partial u_n^2} u_{kl} &= \sum_{k=1}^n \frac{\partial^2 m_{kk}}{\partial u_n^2} u_{kk} = \sum_{k=1}^{n-1} \frac{\partial^2 m_{kk}}{\partial u_n^2} u_{kk} + \frac{\partial^2 m_{nn}}{\partial u_n^2} u_{nn} \\
&= \left[ - (1 + u_n^2)^{-\frac{3}{2}} + 3(1 + u_n^2)^{-\frac{5}{2}} u_n^2 \right] \sum_{k=1}^{n-1} u_{kk} + \left[ - 3(1 + u_n^2)^{-\frac{5}{2}} + 15(1 + u_n^2)^{-\frac{7}{2}} u_n^2 \right] u_{nn} \\
&= \left[ - (1 + u_n^2)^{-1} + 3(1 + u_n^2)^{-2} u_n^2 \right] [u_t - (1 + u_n^2)^{-\frac{3}{2}} u_{nn}] + \left[ - 3(1 + u_n^2)^{-\frac{5}{2}} + 15(1 + u_n^2)^{-\frac{7}{2}} u_n^2 \right] u_{nn} \\
&= - (1 + u_n^2)^{-1} u_t + 3(1 + u_n^2)^{-2} u_n^2 u_t + \left[ - 2(1 + u_n^2)^{-\frac{5}{2}} + 12(1 + u_n^2)^{-\frac{7}{2}} u_n^2 \right] u_{nn}
\end{aligned}$$

For  $j \in B$ , differentiating the equation once in  $x_j$ , we get

$$u_{tj} = \sum_{k=1}^n m_{kk} u_{kkj} + \sum_{k=1}^n \frac{\partial m_{kk}}{\partial u_n} u_{nj} u_{kk} + O(\mathcal{H}_\phi),$$

so

$$\begin{aligned}
\sum_{kl=1}^n m_{kl} u_{klj} &= \sum_{k=1}^n m_{kk} u_{kkj} = u_{tj} - \sum_{k=1}^n \frac{\partial m_{kl}}{\partial u_n} u_{nj} u_{kl} - \sum_{k=1}^n \frac{\partial m_{kl}}{\partial u_j} u_{jj} u_{kl} \\
&= u_{tj} - \sum_{k=1}^n \frac{\partial m_{kk}}{\partial u_n} u_{nj} u_{kk} + O(\mathcal{H}_\phi) \\
&= u_{tj} + (1 + u_n^2)^{-\frac{3}{2}} u_n u_{nj} \sum_{k=1}^{n-1} u_{kk} + 3(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi) \\
&= u_{tj} + (1 + u_n^2)^{-1} u_n u_{nj} [u_t - (1 + u_n^2)^{-\frac{3}{2}} u_{nn}] + 3(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi) \\
&= u_{tj} + (1 + u_n^2)^{-1} u_t u_n u_{nj} + 2(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi),
\end{aligned}$$

and from (3.56),

$$\begin{aligned}
(1 + u_n^2)^{-\frac{3}{2}} u_{nnj} &= u_{tj} - (1 + u_n^2)^{-\frac{1}{2}} \sum_{k=1}^{n-1} u_{kkj} + (1 + u_n^2)^{-1} u_t u_n u_{nj} + 2(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi) \\
&= u_{tj} - (1 + u_n^2)^{-\frac{1}{2}} \sum_{k \in G} u_{kkj} + (1 + u_n^2)^{-1} u_t u_n u_{nj} + 2(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{kl=1}^n \frac{\partial m_{kl}}{\partial u_n} u_{klj} &= \sum_{k=1}^n \frac{\partial m_{kk}}{\partial u_n} u_{kkj} = \sum_{k=1}^{n-1} \frac{\partial m_{kk}}{\partial u_n} u_{kkj} + \frac{\partial m_{nn}}{\partial u_n} u_{nnj} \\
&= -(1+u_n^2)^{-\frac{3}{2}} u_n \sum_{k=1}^{n-1} u_{kkj} - 3(1+u_n^2)^{-\frac{5}{2}} u_n u_{nnj} \\
&= -(1+u_n^2)^{-\frac{3}{2}} u_n \sum_{k \in G} u_{kkj} + O(\mathcal{H}_\phi) \\
&\quad - 3(1+u_n^2)^{-1} u_n \left[ u_{tj} - (1+u_n^2)^{-\frac{1}{2}} \sum_{k=1}^{n-1} u_{kkj} + (1+u_n^2)^{-1} u_t u_n u_{nj} + 2(1+u_n^2)^{-\frac{5}{2}} u_n u_{nj} u_{nn} \right] \\
&= -3(1+u_n^2)^{-1} u_n u_{tj} + 2(1+u_n^2)^{-\frac{3}{2}} u_n \sum_{k \in G} u_{kkj} - 3(1+u_n^2)^{-2} u_t u_n^2 u_{nj} \\
&\quad - 6(1+u_n^2)^{-\frac{7}{2}} u_n^2 u_{nj} u_{nn} + O(\mathcal{H}_\phi).
\end{aligned}$$

So

$$\begin{aligned}
Q_j &= 4(1+u_n^2)^{-\frac{3}{2}} u_n \sum_{k \in G} u_{kkj} \cdot u_{jn} u_n^2 + 2(1+u_n^2)^{-\frac{1}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn}]^2 \\
&\quad + 2 \left[ -3(1+u_n^2)^{-1} u_n u_{tj} - 3(1+u_n^2)^{-2} u_t u_n^2 u_{nj} - 6(1+u_n^2)^{-\frac{7}{2}} u_n^2 u_{nj} u_{nn} \right] u_{jn} u_n^2 \\
&\quad + \left[ -(1+u_n^2)^{-1} u_t + 3(1+u_n^2)^{-2} u_n^2 u_t + \left( -2(1+u_n^2)^{-\frac{5}{2}} + 12(1+u_n^2)^{-\frac{7}{2}} u_n^2 \right) u_{nn} \right] u_{jn}^2 u_n^2 \\
&\quad + 2 \left[ -(1+u_n^2)^{-1} u_t u_n - 2(1+u_n^2)^{-\frac{5}{2}} u_n u_{nn} \right] u_n u_{jn}^2 \\
&\quad + 6u_n u_{jn} \left[ u_{tj} + (1+u_n^2)^{-1} u_t u_n u_{nj} + 2(1+u_n^2)^{-\frac{5}{2}} u_n u_{nj} u_{nn} \right] - 6u_{jn}^2 u_t \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1, \alpha \neq i}^n \frac{1}{u_{ii}} m_{\alpha\alpha} [u_n u_{i j \alpha} - 2u_{i\alpha} u_{jn}]^2 + O(\mathcal{H}_\phi) \\
&= 4(1+u_n^2)^{-\frac{3}{2}} u_n \sum_{k \in G} u_{kkj} \cdot u_{jn} u_n^2 + 2(1+u_n^2)^{-\frac{1}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn}]^2 \\
&\quad - 6(1+u_n^2)^{-1} u_n^2 \cdot u_n u_{tj} u_{jn} - 3(1+u_n^2)^{-2} u_n^4 \cdot u_t u_n^2 u_{nj} + 3(1+u_n^2)^{-1} u_n^2 \cdot u_t u_n^2 u_{nj} \\
&\quad + 6(1+u_n^2)^{-\frac{5}{2}} u_n^2 \cdot u_{nn} u_{jn}^2 + 6u_n u_{jn} u_{tj} - 6u_{jn}^2 u_t \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1, \alpha \neq i}^n \frac{1}{u_{ii}} m_{\alpha\alpha} [u_n u_{i j \alpha} - 2u_{i\alpha} u_{jn}]^2 + O(\mathcal{H}_\phi)
\end{aligned}$$

where

$$\begin{aligned}
& 4(1+u_n^2)^{-\frac{3}{2}}u_n \sum_{k \in G} u_{kkj} \cdot u_{jn} u_n^2 + 2(1+u_n^2)^{-\frac{1}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn}]^2 \\
&= 2(1+u_n^2)^{-\frac{1}{2}} \left( \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn}]^2 + 2(1+u_n^2)^{-1} u_n^2 u_{jn} \sum_{k \in G} [u_n u_{kkj} - 2u_{kk} u_{jn} + 2u_{kk} u_{jn}] \right) \\
&= 2(1+u_n^2)^{-\frac{1}{2}} \left( \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1+u_n^2)^{-1} u_n^2 u_{jn} u_{kk}]^2 - \sum_{k \in G} \frac{1}{u_{kk}} [(1+u_n^2)^{-1} u_n^2 u_{jn} u_{kk}]^2 \right) \\
&\quad + 8(1+u_n^2)^{-\frac{3}{2}} u_n^2 u_{jn}^2 \sum_{k \in G} u_{kk} \\
&= 2(1+u_n^2)^{-\frac{1}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1+u_n^2)^{-1} u_n^2 u_{jn} u_{kk}]^2 \\
&\quad - 2(1+u_n^2)^{-\frac{5}{2}} u_n^4 u_{jn}^2 \sum_{k \in G} u_{kk} + 8(1+u_n^2)^{-\frac{3}{2}} u_n^2 u_{jn}^2 \sum_{k \in G} u_{kk} \\
&= 2(1+u_n^2)^{-\frac{1}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1+u_n^2)^{-1} u_n^2 u_{jn} u_{kk}]^2 \\
&\quad - 2(1+u_n^2)^{-2} u_n^4 u_{jn}^2 [u_t - (1+u_n^2)^{-\frac{3}{2}} u_{nn}] + 8(1+u_n^2)^{-1} u_n^2 u_{jn}^2 [u_t - (1+u_n^2)^{-\frac{3}{2}} u_{nn}] + O(\mathcal{H}_\phi) \\
&= 2(1+u_n^2)^{-\frac{1}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1+u_n^2)^{-1} u_n^2 u_{jn} u_{kk}]^2 + O(\mathcal{H}_\phi) \\
&\quad - 2(1+u_n^2)^{-2} u_n^4 \cdot u_t u_{jn}^2 + 8(1+u_n^2)^{-1} u_n^2 \cdot u_t u_{jn}^2 + [2(1+u_n^2)^{-\frac{7}{2}} u_n^4 - 8(1+u_n^2)^{-\frac{5}{2}} u_n^2] u_{nn} u_{jn}^2
\end{aligned}$$

So we can get

$$\begin{aligned}
Q_j - 2u_n u_{jn} u_{tj} &= 2(1+u_n^2)^{-\frac{1}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1+u_n^2)^{-1} u_n^2 u_{jn} u_{kk}]^2 \\
&\quad - 6(1+u_n^2)^{-1} u_n^2 \cdot u_n u_{tj} u_{jn} - 5(1+u_n^2)^{-2} u_n^4 \cdot u_t u_{nj}^2 + 11(1+u_n^2)^{-1} u_n^2 \cdot u_t u_{nj}^2 \\
&\quad + [2(1+u_n^2)^{-\frac{7}{2}} u_n^4 - 2(1+u_n^2)^{-\frac{5}{2}} u_n^2] u_{nn} u_{jn}^2 + 4u_n u_{jn} u_{tj} - 6u_{jn}^2 u_t \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1, \alpha \neq i}^n \frac{1}{u_{ii}} m_{\alpha\alpha} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}]^2 + O(\mathcal{H}_\phi)
\end{aligned}$$

Hence

$$\begin{aligned}
Q_j - 2u_n u_{jn} u_{tj} &\leq -6(1 + u_n^2)^{-1} u_n^2 \cdot u_n u_{tj} u_{jn} - 5(1 + u_n^2)^{-2} u_n^4 \cdot u_t u_{nj}^2 + 11(1 + u_n^2)^{-1} u_n^2 \cdot u_t u_{nj}^2 \\
&\quad + [2(1 + u_n^2)^{-2} u_n^4 - 2(1 + u_n^2)^{-1} u_n^2] \cdot u_t u_{jn}^2 + 4u_n u_{jn} u_{tj} - 6u_{jn}^2 u_t + O(\mathcal{H}_\phi) \\
&= -6(1 + u_n^2)^{-1} u_n^2 \cdot u_{jn} [u_t u_{jn} - \frac{\hat{h}_{jn}}{u_t}] - 3(1 + u_n^2)^{-2} u_n^4 \cdot u_t u_{nj}^2 \\
&\quad + 9(1 + u_n^2)^{-1} u_n^2 \cdot u_t u_{nj}^2 + 4u_{jn} [u_t u_{jn} - \frac{\hat{h}_{jn}}{u_t}] - 6u_{jn}^2 u_t + O(\mathcal{H}_\phi) \\
&= \frac{\hat{h}_{jn}}{u_t} [6(1 + u_n^2)^{-1} u_n^2 - 4] \cdot u_{jn} \\
&\quad + [-2 + 3(1 + u_n^2)^{-1} u_n^2 - 3(1 + u_n^2)^{-2} u_n^4] \cdot u_t u_{nj}^2 + O(\mathcal{H}_\phi) \\
&= \frac{\hat{h}_{jn}}{u_t} [6(1 + u_n^2)^{-1} u_n^2 - 4] \cdot u_{jn} - \frac{5}{4} u_t u_{nj}^2 \\
&\quad - 3[(1 + u_n^2)^{-1} u_n^2 - \frac{1}{2}]^2 \cdot u_t u_{nj}^2 + O(\mathcal{H}_\phi) \\
&= -u_t \left( \frac{\hat{h}_{jn}}{u_t^2} [3(1 + u_n^2)^{-1} u_n^2 - 2] - u_{jn} \right)^2 + u_t [3(1 + u_n^2)^{-1} u_n^2 - 2]^2 \frac{\hat{h}_{jn}^2}{u_t^4} - \frac{1}{4} u_t u_{nj}^2 \\
&\quad - 3[(1 + u_n^2)^{-1} u_n^2 - \frac{1}{2}]^2 \cdot u_t u_{nj}^2 + O(\mathcal{H}_\phi) \\
&\leq u_t [3(1 + u_n^2)^{-1} u_n^2 - 2]^2 \frac{\hat{h}_{jn}^2}{u_t^4} + O(\mathcal{H}_\phi) = O(\mathcal{H}_\phi)
\end{aligned}$$

So we can get

$$\begin{aligned}
m_{\alpha\beta} \phi_{\alpha\beta} - \phi_t &\leq C(\phi + \sum_{i,j \in B} |\nabla a_{ij}|) \\
&\quad - \frac{1}{\sigma_1^3(B)} \sum_{\alpha, \beta=1}^n \sum_{i \in B} m_{\alpha\beta} [\sigma_1(B) a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \alpha}] [\sigma_1(B) a_{ii, \beta} - a_{ii} \sum_{j \in B} a_{jj, \beta}] \\
&\quad - \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta=1}^n \sum_{i \neq j, i, j \in B} m_{\alpha\beta} a_{ij, \alpha} a_{ij, \beta},
\end{aligned} \tag{3.57}$$

Following the proof of Theorem 3.1, we get (3.52).  $\square$

**Remark 3.6.** The constant rank properties ( that is, Corollary 3.2) still holds for the equation (1.13).

**3.5. Constant rank theorem of the spatial fundamental form for the equation (1.14).** In this subsection, we consider the mean curvature parabolic equation, that is

$$u_t = (1 + |\nabla u|^2)^{\frac{3}{2}} \operatorname{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = M_{kl}(\nabla u) u_{kl}, \text{ in } \Omega \times (0, T], \tag{3.58}$$

where

$$M_{kl}(\nabla u) = (1 + |\nabla u|^2) \delta_{kl} - u_k u_l. \tag{3.59}$$

We will establish the constant rank theorem for the spatial second fundamental form  $a$  as follows.

**Theorem 3.7.** Suppose  $u \in C^{3,1}(\Omega \times (0, T])$  is a spacetime quasiconcave to the parabolic equation (3.58) and satisfies (1.4). Then the second fundamental form of spatial level sets  $\Sigma^c = \{x \in \Omega | u(x, t) = c\}$  has the same constant rank in  $\Omega$  for any fixed  $t \in (0, T]$ . Moreover, let  $l(t)$  be the minimal rank of the second fundamental form in  $\Omega$ , then  $l(s) \leq l(t)$  for all  $0 < s \leq t \leq T$ .

PROOF. The proof is similar to the the proof of Theorem 3.1, with some modifications.

Suppose  $a(x, t)$  attains minimal rank  $l$  at some point  $(x_0, t_0) \in \Omega \times (0, T]$ . We may assume  $l \leq n - 2$ , otherwise there is nothing to prove. So there is a small neighborhood  $O \times (t_0 - \delta, t_0]$  of  $(x_0, t_0)$ , such that there are  $l$  "good" eigenvalues of  $(a_{ij})$  which are bounded below by a positive constant, and the other  $n - 1 - l$  "bad" eigenvalues of  $(a_{ij})$  are very small. Denote  $G$  be the index set of these "good" eigenvalues and  $B$  be the index set of "bad" eigenvalues. We will prove the differential inequality

$$(3.60) \quad \sum_{\alpha, \beta=1}^n M_{\alpha\beta} \phi_{\alpha\beta}(x, t) - \phi_t \leq C(\phi + |\nabla \phi|), \quad \forall (x, t) \in O \times (t_0 - \delta, t_0],$$

where  $\phi$  is defined in (3.4) and  $C$  is a positive constant independent of  $\phi$ . Then by the strong maximum principle and the method of continuity, Theorem 3.7 holds.

For any fixed point  $(x, t) \in O \times (t_0 - \delta, t_0]$ , we may express  $(a_{ij})$  in a form of (2.3), by choosing  $e_1, \dots, e_{n-1}, e_n$  such that

$$(3.61) \quad |\nabla u(x, t)| = u_n(x, t) > 0 \text{ and } (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t).$$

Following the proof of Theorem 3.1, we get from (3.21)

$$(3.62) \quad \begin{aligned} & M_{\alpha\beta} \phi_{\alpha\beta} - \phi_t \\ &= u_n^{-3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] (Q_j - 2u_n u_{jn} u_{jt}) \\ &\quad - \frac{1}{\sigma_1^3(B)} \sum_{\alpha, \beta=1}^n \sum_{i \in B} M_{\alpha\beta} [\sigma_1(B) a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \alpha}] [\sigma_1(B) a_{ii, \beta} - a_{ii} \sum_{j \in B} a_{jj, \beta}] \\ &\quad - \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta=1}^n \sum_{i \neq j, i, j \in B} M_{\alpha\beta} a_{ij, \alpha} a_{ij, \beta} + O(\mathcal{H}_\phi), \end{aligned}$$

where

$$\begin{aligned} Q_j &= 2 \sum_{kl=1}^n \frac{\partial M_{kl}}{\partial u_n} u_{klj} u_{jn} u_n^2 + \sum_{kl=1}^n \frac{\partial^2 M_{kl}}{\partial u_n^2} u_{kl} u_{jn}^2 u_n^2 \\ &\quad + 2 \sum_{kl=1}^n \frac{\partial M_{kl}}{\partial u_n} u_{kl} u_n u_{jn}^2 + 6u_n u_{jn} \sum_{kl=1}^n M_{kl} u_{jkl} - 6u_{jn}^2 \sum_{kl=1}^n M_{kl} u_{kl} \\ &\quad + 2 \sum_{i \in G} \sum_{\alpha, \beta=1}^n \frac{1}{u_{ii}} M_{\alpha\beta} [u_n u_{ij, \alpha} - 2u_{i\alpha} u_{jn}] [u_n u_{ij, \beta} - 2u_{i\beta} u_{jn}]. \end{aligned}$$

Under the coordinate (3.61), we get

$$M_{kl} = 0, k \neq l; \quad M_{kk} = 1 + u_n^2, \quad k < n; \quad M_{nn} = 1;$$

$$\frac{\partial M_{kl}}{\partial u_n} = 0, k \neq l; \quad \frac{\partial M_{kk}}{\partial u_n} = 2u_n, k < n; \quad \frac{\partial M_{nn}}{\partial u_n} = 0;$$

and

$$\frac{\partial^2 M_{kl}}{\partial u_n^2} = 0, k \neq l; \quad \frac{\partial^2 M_{kk}}{\partial u_n^2} = 2, k < n; \quad \frac{\partial^2 M_{nn}}{\partial u_n^2} = 0.$$

From (3.9), (3.11)

$$(3.63) \quad u_{kk} = O(\mathcal{H}_\phi), \quad u_{kkj} = O(\mathcal{H}_\phi), \quad \forall k \in B, j \in B.$$

From (3.13), we get

$$(3.64) \quad \hat{h}_{jn}^2 = O(\mathcal{H}_\phi), \quad \forall j \in B.$$

From the equation (3.58), we know

$$u_t = \sum_{k=1}^n M_{kk} u_{kk} = (1 + u_n^2) \sum_{k=1}^{n-1} u_{kk} + u_{nn},$$

so we get

$$(1 + u_n^2) \sum_{k=1}^{n-1} u_{kk} = u_t - u_{nn},$$

and by  $u_{kk} \leq 0$  for  $k < n$ , it yields

$$u_{nn} \geq u_t.$$

Hence we can get

$$\begin{aligned} \sum_{kl=1}^n \frac{\partial M_{kl}}{\partial u_n} u_{kl} &= \sum_{k=1}^n \frac{\partial M_{kk}}{\partial u_n} u_{kk} = \sum_{k=1}^{n-1} \frac{\partial M_{kk}}{\partial u_n} u_{kk} + \frac{\partial M_{nn}}{\partial u_n} u_{nn} \\ &= 2u_n \sum_{k=1}^{n-1} u_{kk} \\ &= 2u_n(1 + u_n^2)^{-1} [u_t - u_{nn}] \\ &= 2(1 + u_n^2)^{-1} u_t u_n - 2(1 + u_n^2)^{-1} u_n u_{nn}. \end{aligned}$$

and

$$\begin{aligned} \sum_{kl=1}^n \frac{\partial^2 M_{kl}}{\partial u_n^2} u_{kl} &= \sum_{k=1}^n \frac{\partial^2 M_{kk}}{\partial u_n^2} u_{kk} = \sum_{k=1}^{n-1} \frac{\partial^2 M_{kk}}{\partial u_n^2} u_{kk} + \frac{\partial^2 M_{nn}}{\partial u_n^2} u_{nn} \\ &= 2 \sum_{k=1}^{n-1} u_{kk} = 2(1 + u_n^2)^{-1} [u_t - u_{nn}] \\ &= 2(1 + u_n^2)^{-1} u_t - 2(1 + u_n^2)^{-1} u_{nn} \end{aligned}$$

For  $j \in B$ , differentiating the equation once in  $x_j$ , we get

$$\begin{aligned} u_{tj} &= \sum_{k=1}^n M_{kk} u_{kkj} + \sum_{kl=1}^n \frac{\partial M_{kl}}{\partial u_p} u_{pj} u_{kl} \\ &= \sum_{k=1}^n M_{kk} u_{kkj} + \sum_{kl=1}^n \frac{\partial M_{kl}}{\partial u_j} u_{jj} u_{kl} + \sum_{kl=1}^n \frac{\partial M_{kl}}{\partial u_n} u_{nj} u_{kl} \\ &= \sum_{k=1}^n M_{kk} u_{kkj} + \sum_{k=1}^n \frac{\partial M_{kk}}{\partial u_n} u_{nj} u_{kk} + O(\mathcal{H}_\phi), \end{aligned}$$

so

$$\begin{aligned} \sum_{kl=1}^n M_{kl} u_{klj} &= \sum_{k=1}^n M_{kk} u_{kkj} \\ &= u_{tj} - \sum_{k=1}^n \frac{\partial M_{kk}}{\partial u_n} u_{nj} u_{kk} + O(\mathcal{H}_\phi) \\ &= u_{tj} - 2u_n u_{nj} \sum_{k=1}^{n-1} u_{kk} + O(\mathcal{H}_\phi) \\ &= u_{tj} - 2u_n u_{nj} (1 + u_n^2)^{-1} [u_t - u_{nn}] + O(\mathcal{H}_\phi) \\ &= u_{tj} - 2(1 + u_n^2)^{-1} u_t u_n u_{nj} + 2(1 + u_n^2)^{-1} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi) \end{aligned}$$

Hence from (3.63),

$$\begin{aligned} \sum_{kl=1}^n \frac{\partial M_{kl}}{\partial u_n} u_{klj} &= \sum_{k=1}^n \frac{\partial M_{kk}}{\partial u_n} u_{kkj} = \sum_{k=1}^{n-1} \frac{\partial M_{kk}}{\partial u_n} u_{kkj} + \frac{\partial M_{nn}}{\partial u_n} u_{nnj} \\ &= 2u_n \sum_{k=1}^{n-1} u_{kkj} \\ &= 2u_n \sum_{k \in G} u_{kkj} + O(\mathcal{H}_\phi) \end{aligned}$$

So

$$\begin{aligned} Q_j &= 4u_n \sum_{k \in G} u_{kkj} \cdot u_{jn} u_n^2 + 2(1 + u_n^2)^{-1} [u_t - u_{nn}] u_n^2 u_{jn}^2 + 4u_n (1 + u_n^2)^{-1} [u_t - u_{nn}] u_n u_{jn}^2 \\ &\quad + 6u_n u_{jn} [u_{tj} - 2(1 + u_n^2)^{-1} u_t u_n u_{nj} + 2(1 + u_n^2)^{-1} u_n u_{nj} u_{nn}] - 6u_{jn}^2 u_t \\ &\quad + 2(1 + u_n^2) \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn}]^2 + 2 \sum_{i \in G} \sum_{\alpha=1, \alpha \neq i}^n \frac{1}{u_{ii}} M_{\alpha\alpha} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}]^2 + O(\mathcal{H}_\phi) \\ &= 4u_n \sum_{k \in G} u_{kkj} \cdot u_{jn} u_n^2 + 2(1 + u_n^2) \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn}]^2 \\ &\quad - 6(1 + u_n^2)^{-1} u_n^2 \cdot u_{nj}^2 [u_t - u_{nn}] + 6u_n u_{jn} u_{tj} - 6u_{jn}^2 u_t \\ &\quad + 2 \sum_{i \in G} \sum_{\alpha=1, \alpha \neq i}^n \frac{1}{u_{ii}} M_{\alpha\alpha} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}]^2 + O(\mathcal{H}_\phi) \end{aligned}$$

where

$$\begin{aligned}
& 4u_n \sum_{k \in G} u_{kkj} \cdot u_{jn} u_n^2 + 2(1 + u_n^2) \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn}]^2 \\
&= 2(1 + u_n^2) \left( \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn}]^2 + 2(1 + u_n^2)^{-1} u_n^2 u_{jn} \sum_{k \in G} [u_n u_{kkj} - 2u_{kk} u_{jn} + 2u_{kk} u_{jn}] \right) \\
&= 2(1 + u_n^2) \left( \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1 + u_n^2)^{-1} u_n^2 u_{jn} u_{kk}]^2 - \sum_{k \in G} \frac{1}{u_{kk}} [(1 + u_n^2)^{-1} u_n^2 u_{jn} u_{kk}]^2 \right) \\
&\quad + 8u_n^2 u_{jn}^2 \sum_{k \in G} u_{kk} \\
&= 2(1 + u_n^2) \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1 + u_n^2)^{-1} u_n^2 u_{jn} u_{kk}]^2 \\
&\quad - 2(1 + u_n^2)^{-1} u_n^4 u_{jn}^2 \sum_{k \in G} u_{kk} + 8u_n^2 u_{jn}^2 \sum_{k \in G} u_{kk} \\
&= 2(1 + u_n^2) \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1 + u_n^2)^{-1} u_n^2 u_{jn} u_{kk}]^2 \\
&\quad - 2(1 + u_n^2)^{-2} u_n^4 \cdot u_{jn}^2 [u_t - u_{nn}] + 8(1 + u_n^2)^{-1} u_n^2 \cdot u_{jn}^2 [u_t - u_{nn}] + O(\mathcal{H}_\phi)
\end{aligned}$$

So we can get

$$\begin{aligned}
Q_j - 2u_n u_{jn} u_{tj} &= 2(1 + u_n^2) \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1 + u_n^2)^{-1} u_n^2 u_{jn} u_{kk}]^2 \\
&\quad + [2(1 + u_n^2)^{-1} u_n^2 - 2(1 + u_n^2)^{-2} u_n^4] u_{jn}^2 [u_t - u_{nn}] + 4u_n u_{jn} u_{tj} - 6u_{jn}^2 u_t \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1, \alpha \neq i}^n \frac{1}{u_{ii}} m_{\alpha\alpha} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}]^2 + O(\mathcal{H}_\phi) \\
&\leq 4u_n u_{jn} u_{tj} - 6u_{jn}^2 u_t + O(\mathcal{H}_\phi) \\
&= 4u_{jn} [u_t u_{jn} - \frac{\hat{h}_{jn}}{u_t}] - 6u_{jn}^2 u_t + O(\mathcal{H}_\phi) \\
&= -2u_t [u_{jn} + \frac{\hat{h}_{jn}}{u_t^2}]^2 + 2\frac{\hat{h}_{jn}^2}{u_t^3} + O(\mathcal{H}_\phi) \\
&\leq O(\mathcal{H}_\phi)
\end{aligned}$$

So we can get

$$\begin{aligned}
M_{\alpha\beta} \phi_{\alpha\beta} - \phi_t &\leq C(\phi + \sum_{i,j \in B} |\nabla a_{ij}|) \\
&\quad - \frac{1}{\sigma_1^3(B)} \sum_{\alpha, \beta=1}^n \sum_{i \in B} m_{\alpha\beta} [\sigma_1(B) a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \alpha}] [\sigma_1(B) a_{ii, \beta} - a_{ii} \sum_{j \in B} a_{jj, \beta}] \\
&\quad - \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta=1}^n \sum_{i \neq j, i, j \in B} m_{\alpha\beta} a_{ij, \alpha} a_{ij, \beta},
\end{aligned} \tag{3.65}$$

Following the proof of Theorem 3.1, we get (3.60).  $\square$



**Remark 3.8.** The constant rank properties ( that is, Corollary 3.2) still holds for the equation (1.14).

#### 4. CONSTANT RANK THEOREM OF THE SPACETIME SECOND FUNDAMENTAL FORM

In this section, we start to consider the spacetime level sets  $\hat{\Sigma}^c = \{(x, t) \in \Omega \times (0, T) | u(x, t) = c\}$ , and as in Section 2, the Weingarten tensor is

$$(4.1) \quad \hat{a}_{\alpha\beta} = -\frac{|u_t|}{|Du|u_t^3} \hat{A}_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n,$$

where

$$(4.2) \quad \hat{A}_{\alpha\beta} = \hat{h}_{\alpha\beta} - \frac{u_\alpha u_\gamma \hat{h}_{\beta\gamma}}{\hat{W}(1 + \hat{W})u_t^2} - \frac{u_\beta u_\gamma \hat{h}_{\alpha\gamma}}{\hat{W}(1 + \hat{W})u_t^2} + \frac{u_\alpha u_\beta u_\gamma u_\eta \hat{h}_{\gamma\eta}}{\hat{W}^2(1 + \hat{W})^2 u_t^4}, \quad \hat{W} = \frac{|Du|}{|u_t|}.$$

Suppose  $\hat{a}(x, t) = (\hat{a}_{\alpha\beta})_{n \times n}$  attains the minimal rank  $l$  at some point  $(x_0, t_0) \in \Omega \times (0, T]$ . We may assume  $l \leq n - 1$ , otherwise there is nothing to prove. At  $(x_0, t_0)$ , we may choose  $e_1, \dots, e_{n-1}, e_n$  such that

$$(4.3) \quad |\nabla u(x_0, t_0)| = u_n(x_0, t_0) > 0 \quad \text{and} \quad (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x_0, t_0).$$

Without loss of generality we assume  $u_{11} \leq u_{22} \leq \dots \leq u_{n-1, n-1}$ . So, at  $(x_0, t_0)$ , from (4.1), we have the matrix  $(\hat{a}_{ij})_{1 \leq i, j \leq n-1}$  is also diagonal, and  $\hat{a}_{11} \geq \hat{a}_{22} \geq \dots \geq \hat{a}_{n-1, n-1}$ . From lemma 2.5, there is a positive constant  $C_0$  such that at  $(x_0, t_0)$

CASE 1:

$$\begin{aligned} \hat{a}_{11} &\geq \dots \geq \hat{a}_{l-1, l-1} \geq C_0, \quad \hat{a}_{ll} = \dots = \hat{a}_{n-1, n-1} = 0, \\ \hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{in}^2}{\hat{a}_{ii}} &\geq C_0, \quad \hat{a}_{in} = 0, \quad l \leq i \leq n-1. \end{aligned}$$

CASE 2:

$$\begin{aligned} \hat{a}_{11} &\geq \dots \geq \hat{a}_{ll} \geq C_0, \quad \hat{a}_{l+1, l+1} = \dots = \hat{a}_{n-1, n-1} = 0, \\ \hat{a}_{nn} - \sum_{i=1}^l \frac{\hat{a}_{in}^2}{\hat{a}_{ii}} &\geq C_0, \quad \hat{a}_{in} = 0, \quad l+1 \leq i \leq n-1. \end{aligned}$$

**4.1. CASE 1.** In this subsection, we consider CASE 1, that is, at  $(x_0, t_0)$ , we have

$$\begin{aligned} \hat{a}_{11} &\geq \dots \geq \hat{a}_{l-1, l-1} \geq C_0, \quad \hat{a}_{ll} = \dots = \hat{a}_{n-1, n-1} = 0, \\ \hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{in}^2}{\hat{a}_{ii}} &\geq C_0, \quad \hat{a}_{in} = 0, \quad l \leq i \leq n-1. \end{aligned}$$

Then there is a neighborhood  $\mathcal{O} \times (t_0 - \delta, t_0]$  of  $(x_0, t_0)$ , such that for any fixed point  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ , we may choose  $e_1, \dots, e_{n-1}, e_n$  such that

$$(4.4) \quad |\nabla u(x, t)| = u_n(x, t) > 0 \quad \text{and} \quad (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t).$$

Similarly we assume  $u_{11} \leq u_{22} \leq \dots \leq u_{n-1, n-1}$ . So, at  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ , from (4.1), we have the matrix  $(\hat{a}_{ij})_{1 \leq i, j \leq n-1}$  is also diagonal, and  $\hat{a}_{11} \geq \hat{a}_{22} \geq \dots \geq \hat{a}_{n-1, n-1}$ . There is a positive constant  $C_0 > 0$  depending

only on  $\|u\|_{C^4}$  and  $\mathcal{O} \times (t_0 - \delta, t_0]$ , such that

$$\begin{aligned} \hat{a}_{11} &\geq \cdots \geq \hat{a}_{l-l-1} \geq C_0, \\ \hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{in}^2}{\hat{a}_{ii}} &\geq C_0. \end{aligned}$$

for  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ . For convenience we denote  $G = \{1, \dots, l-1\}$  and  $B = \{l, \dots, n-1\}$  be the "good" and "bad" sets of indices respectively.

Since

$$(4.5) \quad \hat{a}_{ij} = \frac{|\nabla u|}{|Du|} a_{ij}, \quad 1 \leq i, j \leq n-1,$$

there is a positive constant  $C > 0$  depending only on  $\|u\|_{C^4}$  and  $\mathcal{O} \times (t_0 - \delta, t_0]$ , such that

$$(4.6) \quad a_{11} \geq \cdots \geq a_{l-l-1} \geq C, \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0],$$

and

$$(4.7) \quad a_{ll}(x_0, t_0) = \cdots = a_{n-1n-1}(x_0, t_0) = 0.$$

So the spatial second fundamental form  $a = (a_{ij})_{n-1 \times n-1}$  attains the minimal rank  $l-1$  at  $(x_0, t_0)$ . From Theorem 3.1, the constant rank theorem holds for the spatial second fundamental form  $a$ . So we can get  $a_{ii} = 0, \forall i \in B$  for any  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ . Furthermore,

$$(4.8) \quad \hat{a}_{ii} = 0, \quad \forall i \in B.$$

We denote  $M = (\hat{a}_{ij})_{1 \leq i, j \leq n-1}$ , so

$$(4.9) \quad \sigma_{l+1}(M) = \sigma_l(M) \equiv 0, \quad \text{for every } (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].$$

Then we have

$$(4.10) \quad 0 \leq \sigma_{l+1}(\hat{a}) \leq \sigma_{l+1}(M) + \hat{a}_{nn} \sigma_l(M) = 0.$$

So

$$(4.11) \quad \sigma_{l+1}(\hat{a}) \equiv 0, \quad \text{for every } (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].$$

By the continuity method, Theorem 1.2 holds under the CASE 1.

**4.2. CASE 2.** In this subsection, we consider CASE 2. From Lemma 2.5,  $\hat{a}(x, t) = (\hat{a}_{\alpha\beta})_{n \times n}$  attains the minimal rank  $l$  at some point  $(x_0, t_0) \in \Omega \times (0, T]$  and at  $(x_0, t_0)$ , we may choose  $e_1, \dots, e_{n-1}, e_n$  such that

$$|\nabla u| = u_n > 0 \quad \text{and} \quad (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x_0, t_0).$$

Then we have

$$\begin{aligned} \hat{a}_{11} &\geq \cdots \geq \hat{a}_{ll} \geq C_0, \quad \hat{a}_{l+l+1} = \cdots = \hat{a}_{n-1n-1} = 0, \\ \hat{a}_{nn} &= \sum_{i=1}^l \frac{\hat{a}_{in}^2}{\hat{a}_{ii}}, \quad \hat{a}_{in} = 0, \quad l+1 \leq i \leq n-1. \end{aligned}$$

Then there is a small enough neighborhood  $\mathcal{O} \times (t_0 - \delta, t_0]$  of  $(x_0, t_0)$ , such that for any fixed point  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ , we may choose  $e_1, \dots, e_{n-1}, e_n$  such that

$$(4.12) \quad |\nabla u(x, t)| = u_n(x, t) > 0 \quad \text{and} \quad (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t).$$

Similarly we assume  $u_{11} \leq u_{22} \leq \dots \leq u_{n-1, n-1}$ . So, at  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ , from (4.1), we have the matrix  $(\hat{a}_{ij})_{1 \leq i, j \leq n-1}$  is also diagonal, and  $\hat{a}_{11} \geq \hat{a}_{22} \geq \dots \geq \hat{a}_{n-1, n-1}$ . There is a positive constant  $C > 0$  depending only on  $\|u\|_{C^4}$  and  $\mathcal{O} \times (t_0 - \delta, t_0]$ , such that

$$\hat{a}_{11} \geq \dots \geq \hat{a}_{ll} \geq C,$$

for all  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ . For convenience we denote  $G = \{1, \dots, l\}$  and  $B = \{l+1, \dots, n-1\}$  be the "good" and "bad" sets of indices respectively. Since

$$\hat{a}_{ij} = \frac{|\nabla u|}{|Du|} a_{ij}, \quad 1 \leq i, j \leq n-1,$$

there is a positive constant  $C > 0$  depending only on  $\|u\|_{C^4}$  and  $\mathcal{O} \times (t_0 - \delta, t_0]$ , such that

$$(4.13) \quad a_{11} \geq \dots \geq a_{l-1, l-1} \geq C, \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0],$$

and

$$a_{ll}(x_0, t_0) = \dots = a_{n-1, n-1}(x_0, t_0) = 0.$$

So the spatial second fundamental form  $a = (a_{ij})_{n-1 \times n-1}$  attains the minimal rank  $l$  at  $(x_0, t_0)$ . From Theorem 3.1, the constant rank theorem holds for the spatial second fundamental form  $a$ . So we can get  $a_{ii} = 0, \forall i \in B$  for any  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ . Furthermore,  $u_{x_i x_i} = 0, \forall i \in B$ .

In order to simplify the calculations, we need a new spacetime coordinate system. For any fixed point  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ ,  $\{e_1, \dots, e_{n-1}, e_n\}$  is the coordinate satisfying (4.12) and (4.13), and  $e_{n+1}$  is the time coordinate. First, by translating  $\{e_n, e_{n+1}\}$ , we get the coordinate  $\{e_1, \dots, e_{n-1}, \hat{e}_n, \hat{e}_{n+1}\}$  with

$$(4.14) \quad z = (z_1, \dots, z_n, z_{n+1}) = (x, t)O,$$

where

$$(4.15) \quad O = (O_{ab})_{n+1 \times n+1} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \cos \theta & \sin \theta \\ & & & -\sin \theta & \cos \theta \end{pmatrix},$$

such that

$$(4.16) \quad u_{z_{n+1}} = |Du| > 0, \quad u_{z_1} = \dots = u_{z_n} = 0, \quad \text{at } (x, t).$$

So from (1.4), we have  $\theta \in (0, \frac{\pi}{2})$ . Now we fix the coordinate  $\{e_{l+1}, \dots, e_{n-1}, \hat{e}_{n+1}\}$  and translate  $\{e_1, \dots, e_l, \hat{e}_n\}$ , and we get the coordinates  $\{\bar{e}_1, \dots, \bar{e}_l, e_{l+1}, \dots, e_{n-1}, \bar{e}_n, \hat{e}_{n+1}\}$  with

$$(4.17) \quad y = (y_1, \dots, y_n, y_{n+1}) = (z_1, \dots, z_n, z_{n+1})T,$$

where

$$(4.18) \quad T = (T_{\alpha\beta})_{n+1 \times n+1} = \begin{pmatrix} T_{11} & \cdots & T_{1l} & & & T_{1n} \\ \vdots & \ddots & \vdots & & & \vdots \\ T_{l1} & \cdots & T_{ll} & & & T_{ln} \\ & & & 1 & & 0 \\ & & & & \ddots & \vdots \\ & & & & & 1 & 0 \\ T_{n1} & \cdots & T_{nl} & 0 & \cdots & 0 & 1 \\ & & & & & & & 1 \end{pmatrix}$$

such that

$$(4.19) \quad (u_{y_\alpha y_\beta})_{1 \leq \alpha, \beta \leq n} \text{ is diagonal at } (x, t).$$

Finally, we get a new spacetime coordinate  $\{\bar{e}_1, \dots, \bar{e}_l, e_{l+1}, \dots, e_{n-1}, \bar{e}_n, \hat{e}_{n+1}\}$  with

$$(4.20) \quad y = (y_1, \dots, y_n, y_{n+1}) = (x, t)P, \quad P = OT$$

such that

$$(4.21) \quad u_{y_{n+1}} = |Du| > 0, \quad u_{y_1} = \dots = u_{y_n} = 0, \quad \text{at } (x, t),$$

$$(4.22) \quad (u_{y_\alpha y_\beta})_{1 \leq \alpha, \beta \leq n} \text{ is diagonal at } (x, t).$$

From (2.13)-(2.15), we get

$$(4.23) \quad \bar{a}_{\alpha\beta} = -\frac{1}{u_{y_{n+1}}} u_{y_\alpha y_\beta}, \quad 1 \leq \alpha, \beta \leq n,$$

Without loss of generality, we can assume  $\frac{\partial^2 u}{\partial y_1 \partial y_1} \leq \dots \leq \frac{\partial^2 u}{\partial y_l \partial y_l} \leq -C < 0$ , where the positive constant  $C > 0$  depending only on  $\|u\|_{C^{3,1}}$ . Then we have

$$(4.24) \quad \bar{a}_{11} \geq \dots \geq \bar{a}_{ll} \geq C.$$

For convenience we denote  $G = \{1, \dots, l\}$  and  $B = \{l+1, \dots, n-1\}$  which mean good terms and bad ones of indices respectively. Without confusion we will also simply denote  $G = \{\bar{a}_{11}, \dots, \bar{a}_{ll}\}$  and  $B = \{\bar{a}_{l+1l+1}, \dots, \bar{a}_{n-1n-1}\}$ . We set

$$(4.25) \quad \phi = \sigma_{l+1}(\bar{a}).$$

In the following, we will prove a differential inequality

$$(4.26) \quad \sum_{i,j=1}^n F^{ij} \phi_{ij} - \phi_t \leq C(\phi + |\nabla_x \phi|) \quad \text{in } \mathcal{O} \times (t_0 - \delta, t_0].$$

In fact, if  $t_0 = T$  and  $(x, t) \in \mathcal{O} \times \{t_0\}$ , we only have (2.30) instead of (2.29) from Lemma 2.9 ( see Remark 2.10 ). So in order to utilizing (2.29), we just prove (4.26) holds for any  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0)$ , with a constant  $C$  independent of  $\text{dist}(\mathcal{O} \times (t_0 - \delta, t_0], \partial(\Omega \times (0, T]))$  and then by a approximation, (4.26) holds for  $t = t_0$ . Then by the strong maximum principle and the method of continuity, we can prove Theorem 1.2 under CASE 2.

For convenience, we will use  $i, j, k, l = 1, \dots, n$  to represent the  $x$  coordinates,  $t$  still the time coordinate, and  $\alpha, \beta, \gamma, \eta = 1, \dots, n+1$  the  $y$  coordinates. And we have

$$(4.27) \quad \frac{\partial y_\alpha}{\partial x_i} = P_{i\alpha}$$

$$(4.28) \quad \frac{\partial y_\alpha}{\partial t} = P_{n+1\alpha}$$

In the following, we always denote

$$\begin{aligned} u_i &= \frac{\partial u}{\partial x_i}, u_t = \frac{\partial u}{\partial t}, u_\alpha = \frac{\partial u}{\partial y_\alpha}, u_{n+1} = \frac{\partial u}{\partial y_{n+1}}, \\ u_{ij} &= \frac{\partial^2 u}{\partial x_i \partial x_j}, u_{it} = \frac{\partial^2 u}{\partial x_i \partial t}, u_{tt} = \frac{\partial^2 u}{\partial t^2}, u_{i\alpha} = \frac{\partial^2 u}{\partial x_i \partial y_\alpha}, \\ u_{\alpha t} &= \frac{\partial^2 u}{\partial y_\alpha \partial t}, u_{\alpha\beta} = \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}, \text{ etc.} \end{aligned}$$

Also, we will use notion  $h = O(\phi)$  if  $|h(x, t)| \leq C(\phi)$  for  $(x, t) \in O \times (t_0 - \delta, t_0)$  with positive constant  $C$  under control, and  $h = O(\phi + |\nabla \phi|)$  has a similar meaning.

From the above discussions, for any  $(x, t) \in O \times (t_0 - \delta, t_0)$  with the coordinate (4.21) and (4.22), we get

$$(4.29) \quad u_{\alpha\alpha} = \frac{\partial^2 u}{\partial y_\alpha \partial y_\alpha} = \frac{\partial^2 u}{\partial x_\alpha \partial x_\alpha} = 0, \quad \forall \alpha \in B.$$

Under above assumptions, we can get

**Proposition 4.1.** *For any  $(x, t) \in O \times (t_0 - \delta, t_0)$  with the coordinate (4.21) and (4.22), we can get*

$$(4.30) \quad \bar{a}_{\alpha\alpha}(x, t) = 0, \quad \alpha \in B.$$

Furthermore, we have from the semipositive definite of  $\bar{a}$ ,

$$(4.31) \quad \bar{a}_{\alpha\beta}(x, t) = 0, \quad \alpha \in B, \beta \in B \cup G,$$

$$(4.32) \quad \bar{a}_{\alpha n}(x, t) = \hat{a}_{n\alpha}(x, t) = 0, \quad \alpha \in B,$$

$$(4.33) \quad D\bar{a}_{\alpha\beta}(x, t) = 0, \quad \alpha \in B, \beta \in B \cup G,$$

$$(4.34) \quad D\bar{a}_{\alpha n}(x, t) = 0, \quad \alpha \in B.$$

PROOF. The proof is directly from the constant rank theorem of  $a$  and Lemma 2.9.

For any  $(x, t) \in O \times (t_0 - \delta, t_0)$  with the coordinate (4.21) and (4.22), we can get from (4.29)

$$\bar{a}_{\alpha\alpha}(x, t) = -\frac{1}{|Du|} u_{\alpha\alpha} = 0, \quad \forall \alpha \in B.$$

From the positive definite of  $\bar{a}$  at  $(x, t)$ , we get

$$\bar{a}_{\alpha\beta}(x, t) = 0, \quad \alpha \in B, \beta \in B \cup G,$$

$$\bar{a}_{\alpha n}(x, t) = 0, \quad \forall \alpha \in B.$$

And from Lemma 2.9 (i.e. Remark 2.10), we get

$$|D\bar{a}_{\alpha\beta}|(x, t) = 0, \quad \alpha \in B, \beta \in B \cup G,$$

$$|D\bar{a}_{\alpha n}|(x, t) = 0, \quad \forall \alpha \in B.$$

So the lemma holds. □

**Lemma 4.2.**

$$(4.35) \quad Du_\alpha = 0, \quad \alpha \in B,$$

$$(4.36) \quad Du_{\alpha\beta} = 0, \quad \alpha \in B, \beta \in B,$$

$$(4.37) \quad Du_{\alpha\beta} = 0, \quad \alpha \in B, \beta \in G \cup \{n\}.$$

PROOF. By the constant rank properties Corollary 3.2, (4.35) holds since the  $y_\alpha$  coordinate is the  $x_\alpha$  coordinate for  $\alpha \in B$ .

By (2.13), (2.14), (4.33) and (4.34), we get for  $\alpha \in B$  and  $\beta = 1, \dots, n$ ,

$$0 = D\bar{a}_{\alpha\beta} = -\frac{|u_{n+1}|}{|Du|u_{n+1}^3} D\bar{A}_{\alpha\beta} = -\frac{|u_{n+1}|}{|Du|u_{n+1}^3} D\bar{h}_{\alpha\beta},$$

so from (2.15), we get

$$\begin{aligned} 0 &= D\bar{h}_{\alpha\beta} = u_{n+1}^2 Du_{\alpha\beta} + 2u_{n+1} Du_{n+1} u_{\alpha\beta} - u_{n+1} u_{\alpha n+1} Du_\beta - u_{n+1} u_{\beta n+1} Du_\alpha \\ &= u_{n+1}^2 Du_{\alpha\beta}, \quad \alpha \in B, \beta = 1, \dots, n. \end{aligned}$$

Hence the lemma holds. □

**Lemma 4.3.**

$$(4.38) \quad u_{y_n y_n} = O(\phi),$$

$$(4.39) \quad u_{x_i y_n} = O(\phi), i < n;$$

$$(4.40) \quad u_{y_n y_n x_i} = O(\phi + |\nabla\phi|), \quad i = 1, \dots, n-1,$$

$$(4.41) \quad u_{y_n y_n x_n} = 2 \frac{1}{u_{y_{n+1}}} u_{y_n x_n} u_{y_n y_{n+1}} + O(\phi + |\nabla\phi|)$$

Proof: In the  $y$  coordinates, we have from (4.30)

$$\phi = \sigma_{l+1}(\bar{a}) = \sigma_l(G)\bar{a}_{nn} \geq 0,$$

so we have

$$(4.42) \quad \bar{a}_{nn} = O(\phi).$$

By (2.13)-(2.15), we have

$$\bar{a}_{nn} = -\frac{|u_{y_{n+1}}|}{|Du|u_{y_{n+1}}^3} \bar{A}_{nn} = -\frac{|u_{y_{n+1}}|}{|Du|u_{y_{n+1}}^3} \bar{h}_{nn} = -\frac{|u_{y_{n+1}}|}{|Du|u_{y_{n+1}}^3} u_{n+1}^2 u_{y_n y_n},$$

so

$$(4.43) \quad \bar{A}_{nn} = O(\phi), \quad \bar{h}_{nn} = O(\phi), \quad u_{y_n y_n} = O(\phi).$$

Taking the first derivatives of  $\phi$  in  $x$ , we have

$$\begin{aligned}
 \phi_i &= \frac{\partial \phi}{\partial x_i} = \sum_{\alpha=1}^n \sigma_l(\bar{a}|\alpha) \bar{a}_{\alpha\alpha,i} \\
 &= \sum_{\alpha \in G} \sigma_l(\bar{a}|\alpha) \bar{a}_{\alpha\alpha,i} + \sum_{\alpha \in B} \sigma_l(\bar{a}|\alpha) \bar{a}_{\alpha\alpha,i} + \sum_{\alpha=n} \sigma_l(\bar{a}|\alpha) \bar{a}_{\alpha\alpha,i} \\
 &= \sigma_l(G) \bar{a}_{nn,i} + O(\phi),
 \end{aligned}
 \tag{4.44}$$

and from (2.13)-(2.15)

$$\begin{aligned}
 \bar{a}_{nn,i} &= \left( -\frac{|u_{n+1}|}{|Du|u_{n+1}|^3} \right)_i \bar{A}_{nn} - \frac{|u_{n+1}|}{|Du|u_{n+1}|^3} \bar{A}_{nn,i} = O(\phi) - \frac{|u_{n+1}|}{|Du|u_{n+1}|^3} \bar{A}_{nn,i} \\
 &= -\frac{|u_{n+1}|}{|Du|u_{n+1}|^3} \bar{h}_{nn,i} + O(\phi) \\
 &= -\frac{|u_{n+1}|}{|Du|u_{n+1}|^3} [u_{y_n y_n}^2 u_{y_n y_n x_i} - 2u_{y_{n+1}} u_{y_n x_i} u_{y_n y_{n+1}}] + O(\phi),
 \end{aligned}
 \tag{4.45}$$

so

$$\bar{a}_{nn,i} = O(\phi + |\nabla \phi|), \quad \bar{A}_{nn,i} = O(\phi + |\nabla \phi|), \quad \bar{h}_{nn,i} = O(\phi + |\nabla \phi|),
 \tag{4.46}$$

and

$$u_{y_n y_n x_i} = 2 \frac{1}{u_{y_{n+1}}} u_{y_n x_i} u_{y_n y_{n+1}} + O(\phi + |\nabla \phi|).
 \tag{4.47}$$

It is easy to know for  $i = 1, \dots, n-1$ ,

$$\begin{aligned}
 u_{y_n x_i} &= u_{y_n y_\alpha} \frac{\partial y_\alpha}{\partial x_i} = u_{y_n y_n} \frac{\partial y_n}{\partial x_i} + u_{y_n y_{n+1}} \frac{\partial y_{n+1}}{\partial x_i} = O(\phi) + u_{y_n y_{n+1}} \frac{\partial y_{n+1}}{\partial x_i} \\
 &= u_{y_n y_{n+1}} \frac{\partial z_{n+1}}{\partial x_i} + O(\phi) = 0 + O(\phi).
 \end{aligned}
 \tag{4.48}$$

Hence the lemma holds from (4.47) and (4.48).  $\square$

**Lemma 4.4.**

$$u_{y_{n+1}} u_{x_n y_n} = u_{x_n} u_{y_n y_{n+1}} + O(\phi),
 \tag{4.49}$$

$$u_{y_{n+1}} u_{y_n t} = u_t u_{y_n y_{n+1}} + O(\phi).
 \tag{4.50}$$

PROOF. By the chain rule, we get

$$u_{x_n} u_{y_n y_{n+1}} = u_{y_\alpha} \frac{\partial y_\alpha}{\partial x_n} u_{y_n y_{n+1}} = u_{y_{n+1}} \frac{\partial y_{n+1}}{\partial x_n} u_{y_n y_{n+1}},$$

and

$$u_{y_{n+1}} u_{x_n y_n} = u_{y_{n+1}} u_{y_\alpha y_n} \frac{\partial y_\alpha}{\partial x_n} = u_{y_{n+1}} u_{y_n y_n} \frac{\partial y_n}{\partial x_n} + u_{y_{n+1}} u_{y_{n+1} y_n} \frac{\partial y_{n+1}}{\partial x_n} = O(\phi) + u_{y_{n+1}} u_{y_{n+1} y_n} \frac{\partial y_{n+1}}{\partial x_n}.$$

so (4.49) holds.

Similarly, we have

$$u_t u_{y_n y_{n+1}} = u_{y_\alpha} \frac{\partial y_\alpha}{\partial t} u_{y_n y_{n+1}} = u_{y_{n+1}} \frac{\partial y_{n+1}}{\partial t} u_{y_n y_{n+1}},$$

and

$$u_{y_{n+1}} u_{y_n t} = u_{y_{n+1}} u_{y_n y_\alpha} \frac{\partial y_\alpha}{\partial t} = u_{y_n y_n} \frac{\partial y_n}{\partial t} + u_{y_{n+1}} u_{y_n y_{n+1}} \frac{\partial y_{n+1}}{\partial t} = O(\phi) + u_{y_{n+1}} u_{y_n y_{n+1}} \frac{\partial y_{n+1}}{\partial t}.$$

so (4.50) holds.  $\square$

**Lemma 4.5.**

$$(4.51) \quad \sum_{kl=1}^n F^{kl} u_{kly_\gamma y_\gamma} = 0, \quad \text{for } \gamma \in B.$$

PROOF. From (3.31) (i.e. the constant rank properties Corollary 3.2) and (4.36), we have for  $\gamma \in B$

$$(4.52) \quad \begin{aligned} \sum_{kl=1}^n F^{kl} u_{kly_\gamma y_\gamma} &= \sum_{kl=1}^n F^{kl} u_{kly_\gamma y_\gamma} - u_{y_\gamma y_\gamma t} \\ &= \sum_{kl=1}^n F^{kl} u_{klx_\gamma x_\gamma} - u_{x_\gamma x_\gamma t} = 2 \sum_{i \in G} \sum_{k,l=1}^n F^{kl} \frac{u_{x_n}^2 u_{ikx_\gamma} u_{ilx_\gamma}}{u_{ii}}. \end{aligned}$$

In fact, for  $i \in G, \gamma \in B$ , we have from (4.36) and (4.37),

$$(4.53) \quad u_{ikx_\gamma} = u_{iky_\gamma} = u_{y_\alpha x_k y_\gamma} \frac{\partial y_\alpha}{\partial x_i} = \sum_{\alpha \leq n} \frac{\partial u_{y_\alpha y_\gamma}}{\partial x_k} \frac{\partial y_\alpha}{\partial x_i} + u_{y_{n+1} x_k y_\gamma} \frac{\partial y_{n+1}}{\partial x_i} = 0 + u_{y_{n+1} k y_\gamma} \frac{\partial z_{n+1}}{\partial x_i} = 0.$$

So (4.51) holds from (4.52) and (4.53).  $\square$

**Lemma 4.6.**

$$(4.54) \quad \sum_{ij=1}^n F^{ij} \phi_{ij} = \sigma_l(G) \sum_{ij=1}^n F^{ij} \bar{a}_{nm,ij} - 2\sigma_l(G) \sum_{ij=1}^n F^{ij} \sum_{\eta \in G} \frac{\bar{a}_{\eta\eta,i} \bar{a}_{\eta\eta,j}}{\bar{a}_{\eta\eta}} + O(\phi + |\nabla_x \phi|).$$

PROOF. Taking the second derivatives of  $\phi$  in  $y$  coordinates, we have

$$(4.55) \quad \begin{aligned} \phi_{\alpha\beta} &= \frac{\partial^2 \phi}{\partial y_\alpha \partial y_\beta} = \sum_{\gamma=1}^n \frac{\partial \sigma_{l+1}}{\partial \bar{a}_{\gamma\gamma}} \bar{a}_{\gamma\gamma,\alpha\beta} + \sum_{\gamma \neq \eta} \frac{\partial^2 \sigma_{l+1}}{\partial \bar{a}_{\gamma\gamma} \partial \bar{a}_{\eta\eta}} \bar{a}_{\gamma\gamma,\alpha} \bar{a}_{\eta\eta,\beta} + \sum_{\gamma \neq \eta} \frac{\partial^2 \sigma_{l+1}}{\partial \bar{a}_{\gamma\eta} \partial \bar{a}_{\eta\gamma}} \bar{a}_{\gamma\eta,\alpha} \bar{a}_{\eta\gamma,\beta} \\ &= \sum_{\gamma=1}^n \sigma_l(\bar{a}|\gamma) \bar{a}_{\gamma\gamma,\alpha\beta} + \sum_{\gamma \neq \eta} \sigma_{l-1}(\bar{a}|\gamma\eta) \bar{a}_{\gamma\gamma,\alpha} \bar{a}_{\eta\eta,\beta} - \sum_{\gamma \neq \eta} \sigma_{l-1}(\bar{a}|\gamma\eta) \bar{a}_{\gamma\eta,\alpha} \bar{a}_{\eta\gamma,\beta}, \end{aligned}$$

where

$$(4.56) \quad \begin{aligned} \sum_{\gamma=1}^n \sigma_l(\bar{a}|\gamma) \bar{a}_{\gamma\gamma,\alpha\beta} &= \sum_{\gamma \in G} \sigma_l(\bar{a}|\gamma) \bar{a}_{\gamma\gamma,\alpha\beta} + \sum_{\gamma \in B} \sigma_l(\bar{a}|\gamma) \bar{a}_{\gamma\gamma,\alpha\beta} + \sum_{\gamma=n} \sigma_l(\bar{a}|\gamma) \bar{a}_{\gamma\gamma,\alpha\beta} \\ &= \sigma_l(G) \sum_{\gamma \in B} \bar{a}_{\gamma\gamma,\alpha\beta} + \sigma_l(G) \bar{a}_{nn,\alpha\beta} + O(\phi) \end{aligned}$$



$$\begin{aligned}
\sum_{\gamma \neq \eta} \sigma_{l-1}(\bar{a}|\gamma\eta) \bar{a}_{\gamma\gamma,\alpha} \bar{a}_{\eta\eta,\beta} &= \sum_{\substack{\gamma\eta \in G \\ \gamma \neq \eta}} \sigma_{l-1}(\bar{a}|\gamma\eta) \bar{a}_{\gamma\gamma,\alpha} \bar{a}_{\eta\eta,\beta} + \sum_{\substack{\gamma=n \\ \eta \in G}} \sigma_{l-1}(\bar{a}|\gamma\eta) \bar{a}_{\gamma\gamma,\alpha} \bar{a}_{\eta\eta,\beta} \\
&\quad + \sum_{\substack{\gamma \in G \\ \eta=n}} \sigma_{l-1}(\bar{a}|\gamma\eta) \bar{a}_{\gamma\gamma,\alpha} \bar{a}_{\eta\eta,\beta} \\
&= O(\phi) + \sum_{\eta \in G} \sigma_{l-1}(G|\eta) \bar{a}_{\eta\eta,\beta} \bar{a}_{nn,\alpha} + \sum_{\gamma \in G} \sigma_{l-1}(G|\gamma) \bar{a}_{\gamma\gamma,\alpha} \bar{a}_{nn,\beta} \\
(4.57) \quad &= \sigma_l(G) \left[ \sum_{\eta \in G} \frac{\bar{a}_{\eta\eta,\beta}}{\bar{a}_{\eta\eta}} \bar{a}_{nn,\alpha} + \sum_{\gamma \in G} \frac{\bar{a}_{\gamma\gamma,\alpha}}{\bar{a}_{\gamma\gamma}} \bar{a}_{nn,\beta} \right] + O(\phi)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\gamma \neq \eta} \sigma_{l-1}(\bar{a}|\gamma\eta) \bar{a}_{\gamma\eta,\alpha} \bar{a}_{\eta\gamma,\beta} &= \sum_{\substack{\gamma\eta \in G \\ \gamma \neq \eta}} \sigma_{l-1}(\bar{a}|\gamma\eta) \bar{a}_{\gamma\eta,\alpha} \bar{a}_{\eta\gamma,\beta} + \sum_{\substack{\gamma=n \\ \eta \in G}} \sigma_{l-1}(\bar{a}|\gamma\eta) \bar{a}_{\gamma\eta,\alpha} \bar{a}_{\eta\gamma,\beta} \\
&\quad + \sum_{\substack{\gamma \in G \\ \eta=n}} \sigma_{l-1}(\bar{a}|\gamma\eta) \bar{a}_{\gamma\eta,\alpha} \bar{a}_{\eta\gamma,\beta} \\
&= O(\phi) + \sum_{\eta \in G} \sigma_{l-1}(G|\eta) \bar{a}_{nn,\alpha} \bar{a}_{\eta n,\beta} + \sum_{\gamma \in G} \sigma_{l-1}(G|\gamma) \bar{a}_{\gamma n,\alpha} \bar{a}_{n\gamma,\beta} \\
(4.58) \quad &= 2\sigma_l(G) \sum_{\eta \in G} \frac{\bar{a}_{nn,\alpha} \bar{a}_{\eta n,\beta}}{\bar{a}_{\eta\eta}} + O(\phi)
\end{aligned}$$

So we have

$$\begin{aligned}
\phi_{\alpha\beta} &= \sigma_l(G) \sum_{\gamma \in B} \bar{a}_{\gamma\gamma,\alpha\beta} + \sigma_l(G) \bar{a}_{nn,\alpha\beta} - 2\sigma_l(G) \sum_{\eta \in G} \frac{\bar{a}_{nn,\alpha} \bar{a}_{\eta n,\beta}}{\bar{a}_{\eta\eta}} \\
(4.59) \quad &\quad + \sigma_l(G) \left[ \sum_{\eta \in G} \frac{\bar{a}_{\eta\eta,\beta}}{\bar{a}_{\eta\eta}} \bar{a}_{nn,\alpha} + \sum_{\gamma \in G} \frac{\bar{a}_{\gamma\gamma,\alpha}}{\bar{a}_{\gamma\gamma}} \bar{a}_{nn,\beta} \right] + O(\phi)
\end{aligned}$$

So we have

$$\begin{aligned}
\sum_{ij=1}^n F^{ij} \phi_{ij} &= \sum_{ij=1}^n F^{ij} \sum_{\alpha\beta=1}^{n+1} P_{i\alpha} P_{j\beta} \phi_{\alpha\beta} \\
&= \sigma_l(G) \sum_{\gamma \in B} \sum_{ij=1}^n F^{ij} \sum_{\alpha\beta=1}^{n+1} P_{i\alpha} P_{j\beta} \bar{a}_{\gamma\gamma, \alpha\beta} + \sigma_l(G) \sum_{ij=1}^n F^{ij} \sum_{\alpha\beta=1}^{n+1} P_{i\alpha} P_{j\beta} \bar{a}_{nn, \alpha\beta} \\
&\quad - 2\sigma_l(G) \sum_{ij=1}^n F^{ij} \sum_{\eta \in G} \frac{[\sum_{\alpha=1}^{n+1} P_{i\alpha} \bar{a}_{nn, \alpha}] [\sum_{\beta=1}^{n+1} P_{j\beta} \bar{a}_{nn, \beta}]}{\bar{a}_{\eta\eta}} \\
&\quad + \sigma_l(G) \sum_{ij=1}^n F^{ij} [\sum_{\eta \in G} \frac{\sum_{\beta=1}^{n+1} P_{j\beta} \bar{a}_{\eta\eta, \beta}}{\bar{a}_{\eta\eta}} \sum_{\alpha=1}^{n+1} P_{i\alpha} \bar{a}_{nn, \alpha} + \sum_{\gamma \in G} \frac{\sum_{\alpha=1}^{n+1} P_{i\alpha} \bar{a}_{\gamma\gamma, \alpha}}{\bar{a}_{\gamma\gamma}} \sum_{\beta=1}^{n+1} P_{j\beta} \bar{a}_{nn, \beta}] + O(\phi) \\
&= \sigma_l(G) \sum_{\gamma \in B} \sum_{ij=1}^n F^{ij} \bar{a}_{\gamma\gamma, ij} + \sigma_l(G) \sum_{ij=1}^n F^{ij} \bar{a}_{nn, ij} - 2\sigma_l(G) \sum_{ij=1}^n F^{ij} \sum_{\eta \in G} \frac{\bar{a}_{nn, i} \bar{a}_{nn, j}}{\bar{a}_{\eta\eta}} \\
&\quad + \sigma_l(G) \sum_{ij=1}^n F^{ij} [\sum_{\eta \in G} \frac{\bar{a}_{\eta\eta, j}}{\bar{a}_{\eta\eta}} \bar{a}_{nn, i} + \sum_{\gamma \in G} \frac{\bar{a}_{\gamma\gamma, i}}{\bar{a}_{\gamma\gamma}} \bar{a}_{nn, j}] + O(\phi) \\
(4.60) \quad &= \sigma_l(G) \sum_{\gamma \in B} \sum_{ij=1}^n F^{ij} \bar{a}_{\gamma\gamma, ij} + \sigma_l(G) \sum_{ij=1}^n F^{ij} \bar{a}_{nn, ij} - 2\sigma_l(G) \sum_{ij=1}^n F^{ij} \sum_{\eta \in G} \frac{\bar{a}_{nn, i} \bar{a}_{nn, j}}{\bar{a}_{\eta\eta}} + O(\phi + |\nabla_x \phi|).
\end{aligned}$$

For  $\gamma \in B$ , we have

$$\bar{a}_{\gamma\gamma, ij} = -\frac{|u_{n+1}|}{|Du|u_{n+1}} \bar{A}_{\gamma\gamma, ij} = -\frac{|u_{n+1}|}{|Du|u_{n+1}} \bar{h}_{\gamma\gamma, ij} = -\frac{|u_{n+1}|}{|Du|u_{n+1}} [u_{y_{n+1}}^2 u_{y_\gamma y_\gamma x_i x_j}],$$

so

$$(4.61) \quad \sigma_l(G) \sum_{\gamma \in B} \sum_{ij=1}^n F^{ij} \bar{a}_{\gamma\gamma, ij} = -\frac{|u_{n+1}|}{|Du|u_{n+1}} \sigma_l(G) \sum_{\gamma \in B} u_{y_{n+1}}^2 \sum_{ij=1}^n F^{ij} u_{y_\gamma y_\gamma x_i x_j} = 0.$$

From (4.60) and (4.61), (4.54) holds.

□

**Lemma 4.7.**

$$(4.62) \quad \phi_t = -u_{y_{n+1}}^{-3} \sigma_l(G) [u_{y_{n+1}}^2 u_{y_n y_n t} - 2u_{y_{n+1}} u_{y_n y_{n+1}} u_{y_n t}] + O(\phi),$$

and

$$\begin{aligned}
\sum_{i,j=1}^n F^{ij} \phi_{ij} &= u_{y_{n+1}}^{-3} \sigma_l(G) [-u_{y_{n+1}}^2 \sum_{i,j=1}^n F^{ij} u_{i j y_n y_n} + 6u_{y_{n+1}} u_{y_n y_{n+1}} \sum_{i,j=1}^n F^{ij} u_{i j y_n} - 6u_{y_n y_{n+1}}^2 \sum_{i,j=1}^n F^{ij} u_{ij}] \\
(4.63) \quad &+ 2u_{y_{n+1}}^{-3} \sigma_l(G) \sum_{\alpha \in G} \sum_{i,j=1}^n F^{ij} \frac{1}{u_{\alpha\alpha}} [u_{y_{n+1}} u_{i j y_n y_\alpha} - 2u_{i y_\alpha} u_{y_n y_{n+1}}] [u_{y_{n+1}} u_{j y_n y_\alpha} - 2u_{j y_\alpha} u_{y_n y_{n+1}}] \\
&+ O(\phi + |\nabla_x \phi|).
\end{aligned}$$

PROOF. Similarly with (4.44), taking the first derivatives of  $\phi$  in  $t$ , we have

$$\begin{aligned}
 \phi_t &= \frac{\partial \phi}{\partial t} = \sum_{\alpha=1}^n \sigma_l(\bar{a}|\alpha) \bar{a}_{\alpha\alpha,t} = \sigma_l(G) \bar{a}_{nn,t} + O(\phi) \\
 &= -\frac{|u_{n+1}|}{|Du|u_{n+1}^3} \sigma_l(G) \bar{h}_{nn,t} + O(\phi) \\
 &= -\frac{1}{u_{n+1}^3} \sigma_l(G) [u_{y_{n+1}}^2 u_{y_n y_n t} - 2u_{y_{n+1}} u_{y_n t} u_{y_n y_{n+1}}] + O(\phi).
 \end{aligned}
 \tag{4.64}$$

In the following, we prove (4.63). In fact, the calculation is similar as in [3] and [16].

It is easy to know

$$\sum_{i,j=1}^n F^{ij} \bar{a}_{nn,ij} - 2 \sum_{i,j=1}^n F^{ij} \sum_{\eta \in G} \frac{\bar{a}_{nn,i} \bar{a}_{nn,j}}{\bar{a}_{nn}} = \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \bar{a}_{nn,\alpha\beta} - 2 \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \sum_{\eta \in G} \frac{\bar{a}_{nn,\alpha} \bar{a}_{nn,\beta}}{\bar{a}_{nn}},
 \tag{4.65}$$

where

$$G^{\alpha\beta} = \sum_{i,j=1}^n F^{ij} P_{i\alpha} P_{j\beta}.
 \tag{4.66}$$

By (2.13)-(2.15), we have

$$\begin{aligned}
 \bar{a}_{nn,\alpha} &= \left(-\frac{|u_{n+1}|}{|Du|u_{n+1}^3}\right)_\alpha \bar{A}_{nn} - \frac{|u_{n+1}|}{|Du|u_{n+1}^3} \bar{A}_{nn,\alpha} = O(\phi) - \frac{|u_{n+1}|}{|Du|u_{n+1}^3} \bar{A}_{nn,\alpha} \\
 &= -\frac{|u_{n+1}|}{|Du|u_{n+1}^3} \bar{h}_{nn,\alpha} + O(\phi) \\
 &= -\frac{1}{u_{n+1}^3} [u_{y_{n+1}}^2 u_{y_n y_n y_\alpha} - 2u_{y_{n+1}} u_{y_n y_\alpha} u_{y_n y_{n+1}}] + O(\phi),
 \end{aligned}
 \tag{4.67}$$

and

$$\begin{aligned}
 \bar{a}_{nn,\alpha\beta} &= \left(-\frac{|u_{n+1}|}{|Du|u_{n+1}^3}\right)_{\alpha\beta} \bar{A}_{nn} + \left(-\frac{|u_{n+1}|}{|Du|u_{n+1}^3}\right)_\alpha \bar{A}_{nn,\beta} + \left(-\frac{|u_{n+1}|}{|Du|u_{n+1}^3}\right)_\beta \bar{A}_{nn,\alpha} - \frac{|u_{n+1}|}{|Du|u_{n+1}^3} \bar{A}_{nn,\alpha\beta} \\
 &= O(\phi) + \left(-\frac{|u_{n+1}|}{|Du|u_{n+1}^3}\right)_\alpha \bar{A}_{nn,\beta} + \left(-\frac{|u_{n+1}|}{|Du|u_{n+1}^3}\right)_\beta \bar{A}_{nn,\alpha} - \frac{|u_{n+1}|}{|Du|u_{n+1}^3} \bar{A}_{nn,\alpha\beta} \\
 &= \left(-\frac{|u_{n+1}|}{|Du|u_{n+1}^3}\right)_\alpha \bar{A}_{nn,\beta} + \left(-\frac{|u_{n+1}|}{|Du|u_{n+1}^3}\right)_\beta \bar{A}_{nn,\alpha} - \frac{1}{u_{n+1}^3} \bar{h}_{nn,\alpha\beta} + O(\phi).
 \end{aligned}
 \tag{4.68}$$

and

$$\begin{aligned}
 \bar{h}_{nn,\alpha\beta} &= u_{n+1}^2 u_{y_n y_n \alpha\beta} + 2u_{n+1} u_{y_{n+1} y_\alpha} u_{y_n y_n y_\beta} + 2u_{n+1} u_{y_{n+1} y_\beta} u_{y_n y_n y_\alpha} + 2u_{y_{n+1} y_{n+1}} u_{y_n y_\alpha} u_{y_n y_\beta} \\
 &\quad - 2u_{y_{n+1}} u_{y_n y_\alpha y_\beta} u_{y_n y_{n+1}} - 2u_{y_{n+1}} u_{y_n y_\alpha} u_{y_n y_{n+1} y_\beta} - 2u_{y_{n+1}} u_{y_n y_\beta} u_{y_n y_{n+1} y_\alpha} \\
 &\quad - 2u_{y_{n+1} y_\beta} u_{y_n y_\alpha} u_{y_n y_{n+1}} - 2u_{y_{n+1} y_\alpha} u_{y_n y_\beta} u_{y_n y_{n+1}} \\
 &= u_{n+1}^2 u_{y_n y_n \alpha\beta} + 2u_{y_{n+1} y_\alpha} u_{y_n y_\beta} u_{y_n y_{n+1}} + 2u_{y_{n+1} y_\beta} u_{y_n y_\alpha} u_{y_n y_{n+1}} + 2u_{y_{n+1} y_{n+1}} u_{y_n y_\alpha} u_{y_n y_\beta} \\
 &\quad - 2u_{y_{n+1}} u_{y_n y_\alpha y_\beta} u_{y_n y_{n+1}} - 2u_{y_{n+1}} u_{y_n y_\alpha} u_{y_n y_{n+1} y_\beta} - 2u_{y_{n+1}} u_{y_n y_\beta} u_{y_n y_{n+1} y_\alpha} \\
 &\quad + 2u_{y_{n+1} y_\alpha} [-u_{y_{n+1}}^2 \bar{a}_{nn,\beta}] + 2u_{y_{n+1} y_\beta} [-u_{y_{n+1}}^2 \bar{a}_{nn,\alpha}] + O(\phi)
 \end{aligned}
 \tag{4.69}$$

Hence,

$$\begin{aligned}
\sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \bar{a}_{nn,\alpha\beta} &= \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \left[ -\frac{1}{u_{n+1}^3} \bar{h}_{nn,\alpha\beta} \right] + O(\phi + |\nabla_x \phi|) \\
&= -\frac{1}{u_{n+1}^3} \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} [u_{n+1}^2 u_{y_n y_n \alpha \beta} + 4u_{y_{n+1} y_\alpha} u_{y_n y_\beta} u_{y_n y_{n+1}} + 2u_{y_{n+1} y_{n+1}} u_{y_n y_\alpha} u_{y_n y_\beta} \\
&\quad - 2u_{y_{n+1}} u_{y_n y_\alpha y_\beta} u_{y_n y_{n+1}} - 4u_{y_{n+1}} u_{y_n y_\alpha} u_{y_n y_{n+1} y_\beta}] + O(\phi + |\nabla_x \phi|).
\end{aligned}
\tag{4.70}$$

where

$$\begin{aligned}
\sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} u_{y_{n+1} y_\alpha} u_{y_n y_\beta} u_{y_n y_{n+1}} &= u_{y_n y_{n+1}}^2 \sum_{\alpha=1}^{n+1} G^{\alpha n+1} u_{y_{n+1} y_\alpha} + O(\phi) \\
&= u_{y_n y_{n+1}}^2 \left( \sum_{\alpha,\beta=1}^{n+1} - \sum_{\beta=1}^n \sum_{\alpha=1}^{n+1} \right) G^{\alpha\beta} u_{\alpha\beta} + O(\phi),
\end{aligned}
\tag{4.71}$$

$$\begin{aligned}
\sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} u_{y_{n+1}} u_{y_n y_\alpha} u_{y_n y_{n+1} y_\beta} &= u_{y_{n+1}} u_{y_n y_{n+1}} \sum_{\beta=1}^{n+1} G^{n+1\beta} u_{y_n y_{n+1} y_\beta} + O(\phi) \\
&= u_{y_{n+1}} u_{y_n y_{n+1}} \left( \sum_{\alpha,\beta=1}^{n+1} - \sum_{\alpha=1}^n \sum_{\beta=1}^{n+1} \right) G^{\alpha\beta} u_{y_n \alpha \beta} + O(\phi),
\end{aligned}
\tag{4.72}$$

and

$$\begin{aligned}
\sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} u_{y_{n+1} y_{n+1}} u_{y_n y_\alpha} u_{y_n y_\beta} &= u_{y_{n+1} y_{n+1}} G^{n+1 n+1} u_{y_n y_{n+1}}^2 + O(\phi) \\
&= u_{y_n y_{n+1}}^2 \sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} u_{y_\alpha y_\beta} - 2u_{y_n y_{n+1}}^2 \sum_{\alpha=1}^n G^{\alpha n+1} u_{y_{n+1} y_\alpha} - u_{y_n y_{n+1}}^2 \sum_{\alpha,\beta=1}^n G^{\alpha\beta} u_{y_\alpha y_\beta} + O(\phi).
\end{aligned}
\tag{4.73}$$

So

$$\begin{aligned}
u_{n+1}^3 \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \bar{a}_{nn,\alpha\beta} &= -u_{n+1}^2 \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} u_{y_n y_n \alpha \beta} + 6u_{y_{n+1}} u_{y_n y_{n+1}} \sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} u_{y_n \alpha \beta} \\
&\quad - 6u_{y_n y_{n+1}}^2 \sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} u_{\alpha\beta} - 4u_{y_{n+1}} u_{y_n y_{n+1}} \sum_{\alpha=1}^n \sum_{\beta=1}^{n+1} G^{\alpha\beta} u_{y_n \alpha \beta} \\
&\quad + 8u_{y_n y_{n+1}}^2 \sum_{\alpha=1}^n G^{\alpha n+1} u_{y_{n+1} y_\alpha} + 6u_{y_n y_{n+1}}^2 \sum_{\alpha,\beta=1}^n G^{\alpha\beta} u_{y_\alpha y_\beta} + O(\phi).
\end{aligned}
\tag{4.74}$$

and

$$\begin{aligned}
 & u_{y_{n+1}} u_{y_n y_{n+1}} \sum_{\alpha=1}^n \sum_{\beta=1}^{n+1} G^{\alpha\beta} u_{y_n \alpha \beta} \\
 = & u_{y_{n+1}} u_{y_n y_{n+1}} \sum_{\beta=1}^{n+1} \left( \sum_{\alpha \in B} G^{\alpha\beta} u_{y_n \alpha \beta} + \sum_{\alpha \in G} G^{\alpha\beta} u_{y_n \alpha \beta} \right) \\
 = & u_{y_{n+1}} u_{y_n y_{n+1}} \sum_{\beta=1}^{n+1} \sum_{\alpha \in G} G^{\alpha\beta} u_{y_n \alpha \beta} \\
 = & u_{y_n y_{n+1}} \sum_{\beta=1}^{n+1} \sum_{\alpha \in G} G^{\alpha\beta} [-u_{y_{n+1}}^2 \bar{a}_{n\alpha\beta} + u_{y_\alpha y_\beta} u_{y_n y_{n+1}} + u_{y_n y_\beta} u_{y_\alpha y_{n+1}}] + O(\phi) \\
 (4.75) \quad = & -u_{y_{n+1}}^2 u_{y_n y_{n+1}} \sum_{\beta=1}^{n+1} \sum_{\alpha \in G} G^{\alpha\beta} \bar{a}_{n\alpha\beta} + u_{y_n y_{n+1}}^2 \sum_{\alpha \in G} G^{\alpha\alpha} u_{y_\alpha y_\alpha} + 2u_{y_n y_{n+1}}^2 \sum_{\alpha \in G} G^{\alpha n+1} u_{y_\alpha y_{n+1}} + O(\phi).
 \end{aligned}$$

(4.74) and (4.75) yield

$$\begin{aligned}
 & u_{n+1}^3 \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \bar{a}_{nn,\alpha\beta} \\
 = & -u_{n+1}^2 \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} u_{y_n y_n \alpha \beta} + 6u_{y_{n+1}} u_{y_n y_{n+1}} \sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} u_{y_n \alpha \beta} - 6u_{y_n y_{n+1}}^2 \sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} u_{\alpha\beta} \\
 (4.76) \quad & + 4u_{y_{n+1}}^2 u_{y_n y_{n+1}} \sum_{\beta=1}^{n+1} \sum_{\alpha \in G} G^{\alpha\beta} \bar{a}_{n\alpha\beta} + 2u_{y_n y_{n+1}}^2 \sum_{\alpha \in G} G^{\alpha\alpha} u_{y_\alpha y_\alpha} + O(\phi).
 \end{aligned}$$

So,

$$\begin{aligned}
 & \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \bar{a}_{nn,\alpha\beta} - 2 \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \sum_{\eta \in G} \frac{\bar{a}_{n\eta,\alpha} \bar{a}_{\eta n,\beta}}{\bar{a}_{\eta\eta}} \\
 = & \frac{1}{u_{y_{n+1}}^3} \left[ -u_{n+1}^2 \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} u_{y_n y_n \alpha \beta} + 6u_{y_{n+1}} u_{y_n y_{n+1}} \sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} u_{y_n \alpha \beta} - 6u_{y_n y_{n+1}}^2 \sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} u_{\alpha\beta} \right] \\
 (4.77) \quad & - 2 \sum_{\eta \in G} \left[ \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \frac{\bar{a}_{n\eta,\alpha} \bar{a}_{\eta n,\beta}}{\bar{a}_{\eta\eta}} - 2 \frac{u_{y_n y_{n+1}}}{u_{y_{n+1}}} \sum_{\beta=1}^{n+1} G^{\eta\beta} \bar{a}_{n\eta,\beta} - \frac{u_{y_n y_{n+1}}^2}{u_{y_{n+1}}^3} G^{\eta\eta} u_{y_\eta y_\eta} \right] + O(\phi).
 \end{aligned}$$

In fact, for any  $\eta \in G$ ,

$$\begin{aligned}
& \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \frac{\bar{a}_{\eta\eta,\alpha} \bar{a}_{\eta\eta,\beta}}{\bar{a}_{\eta\eta}} - 2 \frac{u_{y_n y_{n+1}}}{u_{y_{n+1}}} \sum_{\beta=1}^{n+1} G^{\eta\beta} \bar{a}_{\eta\eta,\beta} - \frac{u_{y_n y_{n+1}}^2}{u_{y_{n+1}}^3} G^{\eta\eta} u_{y_n y_\eta} \\
&= -\frac{1}{u_{y_{n+1}}^3} \left[ \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \frac{\bar{h}_{\eta\eta,\alpha} \bar{h}_{\eta\eta,\beta}}{\bar{h}_{\eta\eta}} - 2 \frac{u_{y_n y_{n+1}}}{u_{y_{n+1}}} \sum_{\beta=1}^{n+1} G^{\eta\beta} \bar{h}_{\eta\eta,\beta} - u_{y_n y_{n+1}}^2 G^{\eta\eta} u_{y_n y_\eta} \right] \\
&= -\frac{1}{u_{y_{n+1}}^3} \left\{ \sum_{\alpha,\beta=1}^n G^{\alpha\beta} \frac{1}{u_{y_{n+1}}^2 u_{\eta\eta}} [u_{y_{n+1}}^2 u_{\eta\eta\alpha} - u_{y_{n+1}} u_{\eta\alpha} u_{y_n y_{n+1}}] [u_{y_{n+1}}^2 u_{\eta\eta\beta} - u_{y_{n+1}} u_{\eta\beta} u_{y_n y_{n+1}}] + O(\phi) \right. \\
&\quad + 2 \sum_{\alpha=1}^n G^{\alpha n+1} \frac{1}{u_{y_{n+1}}^2 u_{\eta\eta}} [u_{y_{n+1}}^2 u_{\eta\eta\alpha} - u_{y_{n+1}} u_{\eta\alpha} u_{y_n y_{n+1}}] [u_{y_{n+1}}^2 u_{\eta\eta n+1} - 2 u_{y_{n+1}} u_{\eta n+1} u_{y_n y_{n+1}}] + O(\phi) \\
&\quad + G^{n+1 n+1} \frac{1}{u_{y_{n+1}}^2 u_{\eta\eta}} [u_{y_{n+1}}^2 u_{\eta\eta n+1} - 2 u_{y_{n+1}} u_{\eta n+1} u_{y_n y_{n+1}}] [u_{y_{n+1}}^2 u_{\eta\eta n+1} - 2 u_{y_{n+1}} u_{\eta n+1} u_{y_n y_{n+1}}] \\
&\quad - 2 \sum_{\alpha=1}^n G^{\alpha\eta} \frac{1}{u_{y_{n+1}}^2 u_{\eta\eta}} [u_{y_{n+1}}^2 u_{\eta\eta\alpha} - u_{y_{n+1}} u_{\eta\alpha} u_{y_n y_{n+1}}] [u_{y_{n+1}} u_{\eta\eta} u_{y_n y_{n+1}}] + O(\phi) \\
&\quad - 2 G^{n+1\eta} \frac{1}{u_{y_{n+1}}^2 u_{\eta\eta}} [u_{y_{n+1}}^2 u_{\eta\eta n+1} - 2 u_{y_{n+1}} u_{\eta n+1} u_{y_n y_{n+1}}] [u_{y_{n+1}} u_{\eta\eta} u_{y_n y_{n+1}}] \\
&\quad \left. + G^{\eta\eta} \frac{1}{u_{y_{n+1}}^2 u_{\eta\eta}} (u_{y_{n+1}} u_{\eta\eta} u_{y_n y_{n+1}})^2 \right\} \\
&= -\frac{1}{u_{y_{n+1}}^3} \sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} \frac{1}{u_{\eta\eta}} [u_{y_{n+1}} u_{\eta\eta\alpha} - 2 u_{\eta\alpha} u_{y_n y_{n+1}}] [u_{y_{n+1}} u_{\eta\eta\beta} - 2 u_{\eta\beta} u_{y_n y_{n+1}}] + O(\phi).
\end{aligned}$$

Then we get

$$\begin{aligned}
& \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \bar{a}_{\eta\eta,\alpha\beta} - 2 \sum_{\alpha\beta=1}^{n+1} G^{\alpha\beta} \sum_{\eta \in G} \frac{\bar{a}_{\eta\eta,\alpha} \bar{a}_{\eta\eta,\beta}}{\bar{a}_{\eta\eta}} \\
&= u_{y_{n+1}}^{-3} \left[ -u_{y_{n+1}}^2 \sum_{i,j=1}^n F^{ij} u_{ij y_n y_n} + 6 u_{y_{n+1}} u_{y_n y_{n+1}} \sum_{i,j=1}^n F^{ij} u_{ij y_n} - 6 u_{y_n y_{n+1}}^2 \sum_{i,j=1}^n F^{ij} u_{ij} \right] \\
&\quad + 2 u_{y_{n+1}}^{-3} \sum_{\alpha \in G} \sum_{i,j=1}^n F^{ij} \frac{1}{u_{\alpha\alpha}} [u_{y_{n+1}} u_{iy_n y_\alpha} - 2 u_{iy_\alpha} u_{y_n y_{n+1}}] [u_{y_{n+1}} u_{jy_n y_\alpha} - 2 u_{jy_\alpha} u_{y_n y_{n+1}}] + O(\phi + |\nabla_x \phi|).
\end{aligned} \tag{4.78}$$

Hence (4.63) holds from (4.65) and (4.78).  $\square$

**Lemma 4.8.**

$$\begin{aligned}
& \sum_{\alpha \in G} \sum_{i,j=1}^n F^{ij} \frac{1}{u_{y_\alpha y_\alpha}} [u_{y_{n+1}} u_{iy_n y_\alpha} - 2 u_{iy_\alpha} u_{y_n y_{n+1}}] [u_{y_{n+1}} u_{jy_n y_\alpha} - 2 u_{jy_\alpha} u_{y_n y_{n+1}}] \\
&\leq \sum_{k \in G} \sum_{i,j=1}^n F^{ij} \frac{1}{u_{x_k x_k}} [u_{y_{n+1}} u_{iy_n x_k} - 2 u_{ix_k} u_{y_n y_{n+1}}] [u_{y_{n+1}} u_{jy_n x_k} - 2 u_{jx_k} u_{y_n y_{n+1}}] + O(\phi + |\nabla_x \phi|).
\end{aligned} \tag{4.79}$$

PROOF. First, we consider a special case:  $F^{ij} = \delta_{ij}$ . That is, we need to prove

$$(4.80) \quad \sum_{\alpha \in G} \sum_{i=1}^n \frac{1}{u_{y_\alpha y_\alpha}} [u_{y_{n+1}} u_{iy_n y_\alpha} - 2u_{iy_\alpha} u_{y_n y_{n+1}}]^2 \leq \sum_{k \in G} \sum_{i=1}^n \frac{1}{u_{x_k x_k}} [u_{y_{n+1}} u_{iy_n x_k} - 2u_{ix_k} u_{y_n y_{n+1}}]^2 + O(\phi + |\nabla_x \phi|).$$

Form (4.35)-(4.37), (4.40) and (4.41), we have

$$(4.81) \quad u_{y_{n+1}} u_{iy_n y_\alpha} - 2u_{iy_\alpha} u_{y_n y_{n+1}} = 0, \quad \alpha \in G, i \in B;$$

$$(4.82) \quad u_{y_{n+1}} u_{iy_n y_\alpha} - 2u_{iy_\alpha} u_{y_n y_{n+1}} = 0, \quad \alpha \in B, i \in G \cup \{n\};$$

$$(4.83) \quad u_{y_{n+1}} u_{iy_n y_n} - 2u_{iy_n} u_{y_n y_{n+1}} = O(\phi + |\nabla_x \phi|), \quad i \in G \cup \{n\}.$$

Since  $\nabla_y^2 u = (u_{y_i y_j})_{1 \leq i, j \leq n}$  is diagonal, by the approximation, we have for  $i \in G \cup \{n\}$

$$(4.84) \quad \sum_{\alpha \in G} \frac{1}{u_{y_\alpha y_\alpha}} [u_{y_{n+1}} u_{iy_n y_\alpha} - 2u_{iy_\alpha} u_{y_n y_{n+1}}]^2 = \lim_{\varepsilon \rightarrow 0+} (u_{y_{n+1}} \nabla_y u_{iy_n} - 2\nabla_y u_i u_{y_n y_{n+1}}) (\nabla_y^2 u - \varepsilon I_n)^{-1} (u_{y_{n+1}} \nabla_y u_{iy_n} - 2\nabla_y u_i u_{y_n y_{n+1}})^T + O(\phi + |\nabla_x \phi|),$$

where  $\varepsilon > 0$  small, and

$$(4.85) \quad \begin{aligned} & (u_{y_{n+1}} \nabla_y u_{iy_n} - 2\nabla_y u_i u_{y_n y_{n+1}}) (\nabla_y^2 u - \varepsilon I_n)^{-1} (u_{y_{n+1}} \nabla_y u_{iy_n} - 2\nabla_y u_i u_{y_n y_{n+1}})^T \\ &= (u_{y_{n+1}} \nabla_z u_{iy_n} - 2\nabla_z u_i u_{y_n y_{n+1}}) T^T (\nabla_y^2 u - \varepsilon I_n)^{-1} T (u_{y_{n+1}} \nabla_z u_{iy_n} - 2\nabla_z u_i u_{y_n y_{n+1}})^T \\ &= (u_{y_{n+1}} \nabla_z u_{iy_n} - 2\nabla_z u_i u_{y_n y_{n+1}}) (\nabla_z^2 u - \varepsilon I_n)^{-1} (u_{y_{n+1}} \nabla_z u_{iy_n} - 2\nabla_z u_i u_{y_n y_{n+1}})^T. \end{aligned}$$

Denote

$$(4.86) \quad C := u_{z_n z_n} - \varepsilon - \sum_{i=1}^l \frac{u_{z_i z_n}^2}{u_{z_i z_i} - \varepsilon} < 0,$$

then

$$\begin{aligned} (\nabla_z^2 u - \varepsilon I_n)^{-1} &= \text{diag}(\frac{1}{u_{z_1 z_1} - \varepsilon}, \dots, \frac{1}{u_{z_l z_l} - \varepsilon}, -\frac{1}{\varepsilon}, \dots, -\frac{1}{\varepsilon}, 0) \\ &\quad + \frac{1}{C} (-\frac{u_{z_1 z_n}}{u_{z_1 z_1} - \varepsilon}, \dots, -\frac{u_{z_l z_n}}{u_{z_l z_l} - \varepsilon}, 0, \dots, 0, 1)^T (-\frac{u_{z_1 z_n}}{u_{z_1 z_1} - \varepsilon}, \dots, -\frac{u_{z_l z_n}}{u_{z_l z_l} - \varepsilon}, 0, \dots, 0, 1) \\ &\leq \text{diag}(\frac{1}{u_{z_1 z_1} - \varepsilon}, \dots, \frac{1}{u_{z_l z_l} - \varepsilon}, 0, \dots, 0, 0). \end{aligned}$$

So

$$(4.87) \quad \begin{aligned} & (u_{y_{n+1}} \nabla_z u_{iy_n} - 2\nabla_z u_i u_{y_n y_{n+1}}) (\nabla_z^2 u - \varepsilon I_n)^{-1} (u_{y_{n+1}} \nabla_z u_{iy_n} - 2\nabla_z u_i u_{y_n y_{n+1}})^T \\ &\leq \sum_{k \in G} \frac{1}{u_{z_k z_k} - \varepsilon} [u_{y_{n+1}} u_{iy_n z_k} - 2u_{iz_k} u_{y_n y_{n+1}}]^2 = \sum_{k \in G} \frac{1}{u_{x_k x_k} - \varepsilon} [u_{y_{n+1}} u_{iy_n x_k} - 2u_{ix_k} u_{y_n y_{n+1}}]^2. \end{aligned}$$

Then we have for  $i \in G \cup \{n\}$

$$\begin{aligned}
 & \sum_{\alpha \in G} \frac{1}{u_{y_\alpha y_\alpha}} [u_{y_{n+1}} u_{iy_n y_\alpha} - 2u_{iy_\alpha} u_{y_n y_{n+1}}]^2 \\
 & \leq \lim_{\varepsilon \rightarrow 0^+} \sum_{k \in G} \frac{1}{u_{x_k x_k} - \varepsilon} [u_{y_{n+1}} u_{iy_n x_k} - 2u_{ix_k} u_{y_n y_{n+1}}]^2 + O(\phi + |\nabla_x \phi|) \\
 (4.88) \quad & = \sum_{k \in G} \frac{1}{u_{x_k x_k}} [u_{y_{n+1}} u_{iy_n x_k} - 2u_{ix_k} u_{y_n y_{n+1}}]^2 + O(\phi + |\nabla_x \phi|).
 \end{aligned}$$

Hence, (4.80) holds from (4.81) and (4.88).

For the general case, the CLAIM also holds following the above proof.  $\square$

**Theorem 4.9.** *Under the assumptions of Theorem 1.2 and the above notations, we have for any fixed point  $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0)$ ,*

$$(4.89) \quad \sum_{i,j=1}^n F^{ij} \phi_{ij} - \phi_t \leq c_0(\phi + |\nabla_x \phi|)$$

PROOF. From (4.62), (4.63) and (4.79),

$$\begin{aligned}
 \sum_{i,j=1}^n F^{ij} \phi_{ij} - \phi_t & \leq u_{y_{n+1}}^{-3} \sigma_l(G) \left[ -u_{y_{n+1}}^2 \left( \sum_{i,j=1}^n F^{ij} u_{ij y_n y_n} - u_{y_n y_{n+1}} \right) - 2u_{y_{n+1}} u_{y_n y_{n+1}} u_{y_n t} \right. \\
 & \quad \left. + 6u_{y_{n+1}} u_{y_n y_{n+1}} \sum_{i,j=1}^n F^{ij} u_{ij y_n} - 6u_{y_n y_{n+1}}^2 \sum_{i,j=1}^n F^{ij} u_{ij} \right] \\
 & \quad + 2u_{y_{n+1}}^{-3} \sigma_l(G) \sum_{k \in G} \sum_{i,j=1}^n F^{ij} \frac{1}{u_{x_k x_k}} [u_{y_{n+1}} u_{iy_n x_k} - 2u_{ix_k} u_{y_n y_{n+1}}] [u_{y_{n+1}} u_{jy_n x_k} - 2u_{jx_k} u_{y_n y_{n+1}}] \\
 (4.90) \quad & + O(\phi + |\nabla_x \phi|).
 \end{aligned}$$



From the equation (1.1), we get

$$\begin{aligned}
u_{y_n y_n t} &= \sum_{ij=1}^n F^{ij} u_{y_n y_n ij} + \sum_{k=1}^n F^{u_k} u_{k y_n y_n} + F^{u, u} u_{y_n y_n} \\
&+ \sum_{ijkl=1}^n F^{ijkl} u_{ij y_n} u_{kl y_n} + 2 \sum_{ijk=1}^n F^{ij, u_k} u_{ij y_n} u_{k y_n} + 2 \sum_{ij=1}^n F^{ij, u} u_{ij y_n} u_{y_n} \\
&+ 2 \sum_{ijk=1}^n F^{ij, x_k} u_{ij y_n} \frac{\partial x_k}{\partial y_n} + 2 \sum_{ij=1}^n F^{ij, t} u_{ij y_n} \frac{\partial t}{\partial y_n} + \sum_{kl=1}^n F^{u_k, u_l} u_{k y_n} u_{l y_n} + 2 \sum_{k=1}^n F^{u_k, u} u_{k y_n} u_{y_n} \\
&+ 2 \sum_{kl=1}^n F^{u_k, x_l} u_{k y_n} \frac{\partial x_l}{\partial y_n} + 2 \sum_{k=1}^n F^{u_k, t} u_{k y_n} \frac{\partial t}{\partial y_n} + F^{u, u} u_{y_n}^2 + 2 \sum_{k=1}^n F^{u, x_k} u_{y_n} \frac{\partial x_k}{\partial y_n} \\
&+ 2 F^{u, t} u_{y_n} \frac{\partial t}{\partial y_n} + \sum_{ik=1}^n F^{x_i, x_k} \frac{\partial x_i}{\partial y_n} \frac{\partial x_k}{\partial y_n} + 2 \sum_{i=1}^n F^{x_i, t} \frac{\partial x_i}{\partial y_n} \frac{\partial t}{\partial y_n} + F^{t, t} \left( \frac{\partial t}{\partial y_n} \right)^2 \\
&= \sum_{ij=1}^n F^{ij} u_{y_n y_n ij} + F^{u_n} u_{x_n y_n y_n} + O(\phi + |\nabla_x \phi|) \\
&+ \sum_{ijkl=1}^n F^{ijkl} u_{ij y_n} u_{kl y_n} + 2 \sum_{ij=1}^n F^{ij, u_n} u_{ij y_n} u_{x_n y_n} + 2 \sum_{ijk=1}^n F^{ij, x_k} u_{ij y_n} \frac{\partial x_k}{\partial y_n} \\
&+ 2 \sum_{ij=1}^n F^{ij, t} u_{ij y_n} \frac{\partial t}{\partial y_n} + F^{u_n, u_n} u_{x_n y_n}^2 + 2 \sum_{l=1}^n F^{u_n, x_l} u_{x_n y_n} \frac{\partial x_l}{\partial y_n} \\
&+ 2 F^{u_n, t} u_{x_n y_n} \frac{\partial t}{\partial y_n} + \sum_{ik=1}^n F^{x_i, x_k} \frac{\partial x_i}{\partial y_n} \frac{\partial x_k}{\partial y_n} + 2 \sum_{i=1}^n F^{x_i, t} \frac{\partial x_i}{\partial y_n} \frac{\partial t}{\partial y_n} + F^{t, t} \left( \frac{\partial t}{\partial y_n} \right)^2,
\end{aligned} \tag{4.91}$$

And from (4.49),

$$u_{x_n} u_{y_n y_{n+1}} = u_{y_{n+1}} u_{x_n y_n} + O(\phi), \tag{4.92}$$

so

$$u_{x_n y_n y_n} = 2 \frac{1}{u_{y_{n+1}}} u_{x_n y_n} u_{y_n y_{n+1}} + O(\phi) = 2 \frac{u_{y_n y_{n+1}}}{u_{y_{n+1}}} u_{x_n y_n} + O(\phi) = 2 \frac{u_{x_n y_n}}{u_{x_n}} u_{x_n y_n} + O(\phi). \tag{4.93}$$

Hence

$$\sum_{i,j=1}^n F^{ij} \phi_{ij} - \phi_t \leq u_{y_{n+1}}^{-3} \sigma_l(G) \left( Q - 2 u_{y_{n+1}} u_{y_n y_{n+1}} u_{y_n t} \right) + O(\phi + |\nabla_x \phi|). \tag{4.94}$$

where

$$\begin{aligned}
Q = & \sum_{ijkl=1}^n F^{ij,kl} u_{ijy_n} u_{kly_n} u_{y_{n+1}}^2 + 2 \sum_{ij=1}^n F^{ij,u_n} u_{ijy_n} u_{x_n y_n} u_{y_{n+1}}^2 + 2 \sum_{ijk=1}^n F^{ij,x_k} u_{ijy_n} \frac{\partial x_k}{\partial y_n} u_{y_{n+1}}^2 \\
& + 2 \sum_{ij=1}^n F^{ij,t} u_{ijy_n} \frac{\partial t}{\partial y_n} u_{y_{n+1}}^2 + F^{u_n, u_n} u_{x_n y_n}^2 u_{y_{n+1}}^2 + 2 \sum_{l=1}^n F^{u_n, x_l} u_{x_n y_n} \frac{\partial x_l}{\partial y_n} u_{y_{n+1}}^2 \\
& + 2 F^{u_n, t} u_{x_n y_n} \frac{\partial t}{\partial y_n} u_{y_{n+1}}^2 + \sum_{ik=1}^n F^{x_i, x_k} \frac{\partial x_i}{\partial y_n} \frac{\partial x_k}{\partial y_n} u_{y_{n+1}}^2 + 2 \sum_{i=1}^n F^{x_i, t} \frac{\partial x_i}{\partial y_n} \frac{\partial t}{\partial y_n} u_{y_{n+1}}^2 + F^{t, t} \left( \frac{\partial t}{\partial y_n} \right)^2 u_{y_{n+1}}^2 \\
(4.95) \quad & + 2 F^{u_n} \frac{1}{u_{x_n}} u_{x_n y_n}^2 u_{y_{n+1}}^2 + 6 \frac{u_{y_{n+1}}^2}{u_{x_n}} u_{y_n x_n} \sum_{ij=1}^n F^{ij} u_{ijy_n} - 6 u_{y_{n+1}}^2 \frac{u_{y_n x_n}^2}{u_{x_n}^2} \sum_{ij=1}^n F^{ij} u_{ij} \\
& + 2 \sum_{k \in G} \sum_{i,j=1}^n F^{ij} \frac{1}{u_{x_k x_k}} [u_{y_{n+1}} u_{iy_n x_k} - 2 u_{ix_k} u_{y_n y_{n+1}}] [u_{y_{n+1}} u_{jy_n x_k} - 2 u_{jx_k} u_{y_n y_{n+1}}],
\end{aligned}$$

From (4.50), we have

$$(4.96) \quad u_{y_{n+1}} u_{y_n y_{n+1}} u_{y_n t} = u_t u_{y_n y_{n+1}}^2 + O(\phi) \geq O(\phi),$$

Set

$$\begin{aligned}
s &= \frac{1}{u_{x_n}}, A_{ij} = s u_{ij} = \frac{u_{ij}}{u_{x_n}}, \theta = (0, \dots, 0, 1), \\
X_{\alpha\beta} &= u_{x_\alpha x_\beta y_n} u_{x_n}, \\
Y &= u_{x_n y_n} u_{x_n}, \\
Z_k &= \frac{\partial x_k}{\partial y_n} u_{x_n}, \\
D &= \frac{\partial t}{\partial y_n} u_{x_n}, \\
V &= ((X_{\alpha\beta}), Y, (Z_i), D) \in \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R};
\end{aligned}$$

then we get

$$X_{i\alpha} = 0, \quad A_{i\alpha} = 0, \quad X_{i\alpha} - 2A_{i\alpha}Y = 0, \quad i \in B.$$

So it yields

$$(4.97) \quad Q = \frac{u_{y_{n+1}}^2}{u_{x_n}^2} Q^*(V, V),$$

where  $Q^*(V, V)$  is defined in (2.23).

From the structural condition (1.3) (i.e. Lemma 2.7),

$$Q^*(V, V) \leq 0.$$

so we get

$$(4.98) \quad Q = \frac{u_{y_{n+1}}^2}{u_{x_n}^2} Q^*(V, V) \leq 0.$$

Hence (4.89) holds from (4.94), (4.96) and (4.98).  $\square$

**4.3. Constant rank theorem of the spacetime fundamental form for the equations (1.12)-(1.14).** Following the proof of Theorem 1.2, we establish the constant rank theorem for the spacetime fundamental form for the equations (1.12)-(1.14) in this subsection as follows.

**Theorem 4.10.** *Suppose  $u \in C^{3,1}(\Omega \times [0, T])$  is a spacetime quasiconcave to the parabolic equation (1.12)-(1.14), and satisfies the condition (1.4). Then the second fundamental form of spatial level sets  $\hat{\Sigma}^c = \{(x, t) \in \Omega \times (0, T) | u(x, t) = c\}$  has the same constant rank in  $\Omega$  for any fixed  $t \in (0, T]$ . Moreover, let  $l(t)$  be the minimal rank of the second fundamental form in  $\Omega$ , then  $l(s) \leq l(t)$  for all  $0 < s \leq t \leq T$ .*

PROOF. The proof is following the proof of Theorem 1.2.

Suppose  $\hat{a}(x, t) = (\hat{a}_{\alpha\beta})_{n \times n}$  attains the minimal rank  $l$  at some point  $(x_0, t_0) \in \Omega \times (0, T]$ . We may assume  $l \leq n - 1$ , otherwise there is nothing to prove. At  $(x_0, t_0)$ , we may choose  $e_1, \dots, e_{n-1}, e_n$  such that

$$(4.99) \quad |\nabla u(x_0, t_0)| = u_n(x_0, t_0) > 0 \quad \text{and} \quad (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x_0, t_0).$$

Without loss of generality we assume  $u_{11} \leq u_{22} \leq \dots \leq u_{n-1, n-1}$ . So, at  $(x_0, t_0)$ , from (4.1), we have the matrix  $(\hat{a}_{ij})_{1 \leq i, j \leq n-1}$  is also diagonal, and  $\hat{a}_{11} \geq \hat{a}_{22} \geq \dots \geq \hat{a}_{n-1, n-1}$ . From lemma 2.5, there is a positive constant  $C_0$  such that at  $(x_0, t_0)$

CASE 1:

$$\begin{aligned} \hat{a}_{11} &\geq \dots \geq \hat{a}_{l-1, l-1} \geq C_0, \quad \hat{a}_{ll} = \dots = \hat{a}_{n-1, n-1} = 0, \\ \hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{in}^2}{\hat{a}_{ii}} &\geq C_0, \quad \hat{a}_{in} = 0, \quad l \leq i \leq n-1. \end{aligned}$$

CASE 2:

$$\begin{aligned} \hat{a}_{11} &\geq \dots \geq \hat{a}_{ll} \geq C_0, \quad \hat{a}_{l+1, l+1} = \dots = \hat{a}_{n-1, n-1} = 0, \\ \hat{a}_{nn} &= \sum_{i=1}^l \frac{\hat{a}_{in}^2}{\hat{a}_{ii}}, \quad \hat{a}_{in} = 0, \quad l+1 \leq i \leq n-1. \end{aligned}$$

For the CASE 1, the theorem holds from Subsection 4.1 and Theorem 3.3, Theorem 3.5, Theorem 3.7.

For the CASE 2, we need to prove the differential inequality (4.89), which is similar to the proof of Theorem 3.3, Theorem 3.5, and Theorem 3.7, with some modifications. In the following, we just prove (4.89) for the equation (1.12). And for the equation (1.13) and (1.14), the proofs follow from the proofs of Theorem 3.5 and Theorem 3.7 with the same modifications.

For the equation (1.12), following the assumptions and notations, we need to prove

$$(4.100) \quad \sum_{\alpha, \beta=1}^n L_{\alpha\beta} \phi_{\alpha\beta}(x, t) - \phi_t \leq C(\phi + |\nabla_x \phi|), \quad \forall (x, t) \in O \times (t_0 - \delta_0, t_0),$$

where  $\phi$  is defined in (4.25) and  $C$  is a positive constant independent of  $\phi$ . Then by a approximation, (4.100) holds for  $t = t_0$ . Then by the strong maximum principle and the method of continuity, we can prove Theorem 4.10 under CASE 2.

For any fixed point  $(x, t) \in O \times (t_0 - \delta, t_0]$ , following the Subsection 4.2, we first choose the space coordinates  $e_1, \dots, e_{n-1}, e_n, e_{n+1}$  still the time coordinate with

$$y = (y_1, \dots, y_n, y_{n+1}) = (x, t)P, \quad P = OT,$$

such that

$$(4.101) \quad |\nabla u(x, t)| = u_n(x, t) > 0 \text{ and } (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t).$$

Finally, we get a new spacetime coordinate  $\{\bar{e}_1, \dots, \bar{e}_l, e_{l+1}, \dots, e_{n-1}, \bar{e}_n, \hat{e}_{n+1}\}$  such that

$$(4.102) \quad u_{y_{n+1}} = |Du| > 0, \quad u_{y_1} = \dots = u_{y_n} = 0, \quad \text{at } (x, t),$$

$$(4.103) \quad (u_{y_\alpha y_\beta})_{1 \leq \alpha, \beta \leq n} \text{ is diagonal at } (x, t).$$

Also we will use  $i, j, k, l = 1, \dots, n$  to represent the  $x$  coordinates,  $t$  still the time coordinate, and  $\alpha, \beta, \gamma, \eta = 1, \dots, n+1$  the  $y$  coordinates.

Following the proof of Theorem 4.9, we get from (4.94)

$$(4.104) \quad \begin{aligned} \sum_{i,j=1}^n L_{ij} \phi_{ij} - \phi_t &\leq u_{y_{n+1}}^{-3} \sigma_l(G) \left( Q - 2u_{y_{n+1}} u_{y_n y_{n+1}} u_{y_n t} \right) + O(\phi + |\nabla_x \phi|) \\ &= u_{y_{n+1}}^{-3} \sigma_l(G) \frac{u_{y_{n+1}}^2}{u_{x_n}^2} \left( Q^*(V, V) - 2u_{x_n} u_{x_n y_n} u_{y_n t} \right) + O(\phi + |\nabla_x \phi|). \end{aligned}$$

where

$$\begin{aligned} Q^*(V, V) &= 2 \sum_{kl=1}^n \frac{\partial L_{kl}}{\partial u_n} u_{kly_n} u_{ny_n} u_n^2 + \sum_{kl=1}^n \frac{\partial^2 L_{kl}}{\partial u_n^2} u_{kl} u_{ny_n}^2 u_n^2 \\ &\quad + 2 \sum_{kl=1}^n \frac{\partial L_{kl}}{\partial u_n} u_{kl} u_n u_{ny_n}^2 + 6u_n u_{ny_n} \sum_{kl=1}^n L_{kl} u_{kly_n} - 6u_{ny_n}^2 \sum_{kl=1}^n L_{kl} u_{kl} \\ &\quad + 2 \sum_{i \in G} \sum_{\alpha, \beta=1}^n \frac{1}{u_{ii}} L_{\alpha\beta} [u_n u_{i\alpha y_n} - 2u_{i\alpha} u_{ny_n}] [u_n u_{i\beta y_n} - 2u_{i\beta} u_{ny_n}]. \end{aligned}$$

Under the coordinate (4.101) ( i.e. (3.38)), we still have (3.40) - (3.46). So from the equation (1.12), we know

$$u_t = L_{kk} u_{kk},$$

and differentiating the equation in  $y_n$ ,

$$(4.105) \quad \begin{aligned} u_{ty_n} &= L_{kk} u_{kky_n} + \frac{\partial L_{kl}}{\partial u_i} u_{iy_n} u_{kl} = L_{kk} u_{kky_n} + \frac{\partial L_{kl}}{\partial u_n} u_{ny_n} u_{kl} + O(\phi) \\ &= L_{kk} u_{kky_n} + (p-2) \frac{L_{kk}}{u_n} u_{ny_n} u_{kk} + O(\phi) \\ &= L_{kk} u_{kky_n} + (p-2) \frac{u_t}{u_n} u_{ny_n} + O(\phi). \end{aligned}$$

So

$$\begin{aligned}
Q^*(V, V) &= 2 \sum_{k=1}^n (p-2) \frac{L_{kk}}{u_n} u_{kky_n} u_{ny_n} u_n^2 + \sum_{k=1}^n (p-2)(p-3) \frac{L_{kk}}{u_n^2} u_{kk} u_{ny_n}^2 u_n^2 \\
&\quad + 2 \sum_{k=1}^n (p-2) \frac{L_{kk}}{u_n} u_{kk} u_n u_{ny_n}^2 + 6 u_n u_{ny_n} \sum_{k=1}^n L_{kk} u_{kky_n} - 6 u_{ny_n}^2 u_t \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{i\alpha y_n} - 2 u_{i\alpha} u_{ny_n}]^2 + O(\phi) \\
&= 2(p-2) [u_{ty_n} - (p-2) \frac{u_t}{u_n} u_{ny_n}] u_{ny_n} u_n + (p-2)(p-3) u_t u_{ny_n}^2 \\
&\quad + 2(p-2) u_t u_{ny_n}^2 + 6 u_n u_{ny_n} [u_{ty_n} - (p-2) \frac{u_t}{u_n} u_{ny_n}] - 6 u_{ny_n}^2 u_t \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{i\alpha y_n} - 2 u_{i\alpha} u_{ny_n}]^2 + O(\phi) \\
&= (2p+2) u_n u_{ny_n} u_{ty_n} - (p^2 + p) u_t u_{ny_n}^2 \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{i\alpha y_n} - 2 u_{i\alpha} u_{ny_n}]^2 + O(\phi).
\end{aligned}$$

Hence from (4.49) and (4.50),

$$\begin{aligned}
Q^*(V, V) - 2 u_n u_{ny_n} u_{ty_n} &= 2 p u_n u_{ny_n} u_{ty_n} - (p^2 + p) u_t u_{ny_n}^2 \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{i\alpha y_n} - 2 u_{i\alpha} u_{ny_n}]^2 + O(\phi) \\
&= 2 p u_{ny_n} [u_t u_{ny_n} + O(\phi)] - (p^2 + p) u_t u_{ny_n}^2 \\
&\quad + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{i\alpha y_n} - 2 u_{i\alpha} u_{ny_n}]^2 + O(\phi) \\
&= -(p^2 - p) u_t u_{ny_n}^2 + 2 \sum_{i \in G} \sum_{\alpha=1}^n \frac{1}{u_{ii}} L_{\alpha\alpha} [u_n u_{i\alpha y_n} - 2 u_{i\alpha} u_{ny_n}]^2 + O(\phi) \\
&\leq C\phi.
\end{aligned}$$

So we get (4.100).  $\square$

## 5. CURVATURE ESTIMATES OF THE SPATIAL LEVEL SETS OF THE SPACETIME QUASICONCAVE SOLUTIONS

In this section, through modifying the proof of Theorem 1.3, we will give a proof of Theorem 1.5, and we prove the second part of Theorem 1.7.

**5.1. Proof of Theorem 1.5.** Suppose that  $u(x, t)$  is a spacetime quasiconcave solution of the equation (1.6), then the spatial level surface  $\Sigma^c = \{x \in \Omega | u(x, t) = c\}$  is convex, that the spatial fundamental form  $a = (a_{ij})_{n-1 \times n-1}$  is semipositive definite.

Set

$$(5.1) \quad \tilde{a} = a - \eta_0 g I, \quad \eta_0 \geq 0, \quad g(x, t) = e^{Au(x, t)},$$

where  $A > 0$  is a constant to be determined. We want to show  $\bar{a}$  is of constant rank. Theorem 1.3 corresponds to the case  $\eta_0 = 0$ . If  $\min\{\kappa^0, \kappa^1\} = 0$ , there is nothing to prove instead of utilizing Theorem 1.3. We will assume  $\min\{\kappa^0, \kappa^1\} > 0$  in the rest of this subsection. Then by Theorem 1.3, the spatial fundamental form  $a = (a_{ij})_{n-1 \times n-1}$  is positive definite in  $\bar{\Omega} \times [0, T]$ . Denote  $\kappa_s(x, t)$  and  $\bar{\kappa}_s(x, t)$  be the minimum eigenvalue of matrix  $a(x, t)$  and  $\bar{a}(x, t)$  respectively. Then  $\kappa_s(x, t)$  has a positive lower bound.

For a positive constant  $A$  to be determined, increasing  $\eta_0$  from 0, such that  $\bar{a}$  is degenerate at some points, i.e.  $\bar{a}$  is semi-positive with the rank is not full. (1.11) follows easily if this happens on the boundary. We want to show that, if the degeneracy happens at an interior point of  $\Omega$ , then  $\bar{a}$  is degenerate in a neighborhood of the interior point with the same rank and we can get a contradiction for a big  $A$ . Then Theorem 1.5 is proved.

Therefore, the main task is to prove the constant rank theorem for  $\bar{a}$ . Suppose  $\bar{a}(x, t)$  attains minimal rank  $l$  at some point  $(x_0, t_0) \in \Omega \times (0, T]$ . We may assume  $l \leq n - 2$ , otherwise there is nothing to prove. And we assume  $u \in C^{3,1}$  and  $u_n > 0$ . So there is a neighborhood  $O \times (t_0 - \delta, t_0]$  of  $(x_0, t_0)$ , such that there are  $l$  "good" eigenvalues of  $(\bar{a}_{ij})$  which are bounded below by a positive constant, and the other  $n - 1 - l$  "bad" eigenvalues of  $(\bar{a}_{ij})$  are very small. Denote  $G$  be the index set of these "good" eigenvalues and  $B$  be the index set of "bad" eigenvalues. And for any fixed point  $(x, t) \in O \times (t_0 - \delta, t_0]$ , we may express  $(\bar{a}_{ij})$  in a form of (5.1) and (2.3), by choosing  $e_1, \dots, e_{n-1}, e_n$  such that

$$(5.2) \quad |\nabla u(x, t)| = u_n(x, t) > 0 \text{ and } (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t).$$

Without loss of generality, we assume  $u_{11} \leq u_{22} \leq \dots \leq u_{n-1, n-1}$ . From (2.2) - (2.4), we have the matrix  $(a_{ij})_{1 \leq i, j \leq n-1}$  is also diagonal, and  $a_{11} \geq a_{22} \geq \dots \geq a_{n-1, n-1}$ , then  $\bar{a}_{11} \geq \bar{a}_{22} \geq \dots \geq \bar{a}_{n-1, n-1}$ . There is a positive constant  $C > 0$  depending only on  $\|u\|_{C^4}$  and  $O \times (t_0 - \delta, t_0]$ , such that  $\bar{a}_{11} \geq \bar{a}_{22} \geq \dots \geq \bar{a}_{ll} > C$  for all  $(x, t) \in O \times (t_0 - \delta, t_0]$ . For convenience we denote  $G = \{1, \dots, l\}$  and  $B = \{l+1, \dots, n-1\}$  be the "good" and "bad" sets of indices respectively. If there is no confusion, we also denote

$$(5.3) \quad G = \{\bar{a}_{11}, \dots, \bar{a}_{ll}\} \text{ and } B = \{\bar{a}_{l+1, l+1}, \dots, \bar{a}_{n-1, n-1}\}.$$

Note that for any  $\delta > 0$ , we may choose  $O \times (t_0 - \delta, t_0]$  small enough such that  $\bar{a}_{jj} < \delta$  for all  $j \in B$  and  $(x, t) \in O \times (t_0 - \delta, t_0]$ .

For each  $(x, t)$ , let  $a = (a_{ij})$  be the symmetric Weingarten tensor of  $\Sigma^{u(x, t)}$ . Set

$$(5.4) \quad p(\bar{a}) = \sigma_{l+1}(\bar{a}_{ij}), \quad q(\bar{a}) = \begin{cases} \frac{\sigma_{l+2}(\bar{a}_{ij})}{\sigma_{l+1}(\bar{a}_{ij})}, & \text{if } \sigma_{l+1}(\bar{a}_{ij}) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.2 is equivalent to say  $p(\bar{a}) \equiv 0$  (defined in (5.4)) in  $O \times (t_0 - \delta, t_0]$ . As in the description of the proof of Theorem 1.3, we should consider the function

$$(5.5) \quad \phi(\bar{a}) = p(\bar{a}) + q(\bar{a}),$$

where  $p$  and  $q$  as in (5.4). We will prove the differential inequality

$$(5.6) \quad \sum_{\alpha, \beta=1}^n F^{\alpha\beta} \phi_{\alpha\beta}(x, t) - \phi_t < C(\phi + |\nabla \phi|), \quad \forall (x, t) \in O \times (t_0 - \delta, t_0],$$

where  $C$  is a positive constant independent of  $\phi$ . Then by the strong maximum principle, we should have

$$\phi(\bar{a}) \equiv 0, \quad \forall (x, t) \in O \times (t_0 - \delta, t_0],$$

but this makes (5.6) to be a equality. Contradiction. So the degeneracy of  $\bar{a}$  cannot happen at an interior point. So Theorem 1.5 holds.

We will use notion  $h = O(f)$  if  $|h(x, t)| \leq Cf(x, t)$  for  $(x, t) \in O \times (t_0 - \delta, t_0]$  with positive constant  $C$  under control.

To get around  $\sigma_{l+1}(\bar{a}_{ij}) = 0$  in  $q(\bar{a})$ , for  $\varepsilon > 0$  sufficiently small, we instead consider

$$(5.7) \quad \phi_\varepsilon(\bar{a}) = \phi(\bar{a}_\varepsilon),$$

where  $a_\varepsilon = \bar{a} + \varepsilon I$ . We will also denote  $G_\varepsilon = \{\bar{a}_{ii} + \varepsilon, i \in G\}$ ,  $B_\varepsilon = \{\bar{a}_{ii} + \varepsilon, i \in B\}$ .

To simplify the notations, we will drop subindex  $\varepsilon$  with the understanding that all the estimates will be independent of  $\varepsilon$ . In this setting, if we pick  $O \times (t_0 - \delta, t_0]$  small enough, there is  $C > 0$  independent of  $\varepsilon$  such that

$$(5.8) \quad \phi(\bar{a}(x, t)) \geq C\varepsilon, \quad \sigma_1(B) \geq C\varepsilon, \quad \text{for all } (x, t) \in O \times (t_0 - \delta, t_0 + \delta].$$

We also denote

$$(5.9) \quad \mathcal{H}_\phi = \sum_{i,j \in B} |\nabla \bar{a}_{ij}| + \phi.$$

For any fixed point  $(x, t) \in O \times (t_0 - \delta, t_0]$ , we choose a coordinate system as in (5.1) so that  $|\nabla u| = u_n > 0$  and the matrix  $(\bar{a}_{ij}(x, t))_{1 \leq i, j \leq n-1}$  is diagonal for and semipositive definite. From the definition of  $\phi$ , we get

$$\phi \geq \sigma_l(G) \sum_{i \in B} \bar{a}_{ii} \geq 0,$$

so

$$(5.10) \quad \bar{a}_{ii} = O(\phi) = O(\mathcal{H}_\phi), \quad \forall i \in B.$$

Then

$$(5.11) \quad a_{ii} = O(\mathcal{H}_\phi) + \eta_0 g, \quad \forall i \in B.$$

And from (5.1) and (2.2) - (2.4), we get

$$a_{ii} = -\frac{h_{ii}}{u_n^3} = -\frac{u_{ii}}{u_n},$$

so

$$(5.12) \quad h_{ii} = O(\mathcal{H}_\phi) + \eta_0 g \cdot O(1), \quad u_{ii} = O(\mathcal{H}_\phi) + \eta_0 g \cdot O(1), \quad \forall i \in B.$$

From the definition of  $a_{ij}$ , and  $u_k = 0$  for  $k = 1, \dots, n-1$ , we can get

$$(5.13) \quad \begin{aligned} a_{ij,\alpha} &= \left( -\frac{|u_n|}{|\nabla u| u_n^3} \right)_\alpha h_{ij} + \left( -\frac{|u_n|}{|\nabla u| u_n^3} \right) h_{ij,\alpha} \\ &= 3u_n^{-4} u_n^2 u_{ij} - u_n^{-3} [u_n^2 u_{ij\alpha} + 2u_n u_{n\alpha} u_{ij} - u_{i\alpha} u_n u_{jn} - u_{j\alpha} u_n u_{in}] \\ &= -u_n^{-2} [u_n u_{ij\alpha} - u_{n\alpha} u_{ij} - u_{i\alpha} u_{jn} - u_{j\alpha} u_{in}], \end{aligned}$$

then

$$\begin{aligned}
 \widetilde{a}_{ij,\alpha} &= a_{ij,\alpha} - \eta_0 g \cdot A \cdot u_\alpha \delta_{ij} \\
 (5.14) \quad &= -u_n^{-2} [u_n u_{ij\alpha} - u_{n\alpha} u_{ij} - u_{i\alpha} u_{jn} - u_{j\alpha} u_{in}] + \eta_0 g \cdot A \cdot O(1).
 \end{aligned}$$

So for  $i, j \in B$ , we get

$$(5.15) \quad u_{ij\alpha} = O(\mathcal{H}_\phi) + \eta_0 g [\cdot A \cdot O(1) + O(1)], \quad \forall \alpha < n,$$

$$(5.16) \quad u_n u_{ijn} = 2u_{in} u_{jn} + O(\mathcal{H}_\phi) + \eta_0 g [\cdot A \cdot O(1) + O(1)], \quad \forall \alpha < n,$$

In fact, from (2.6) -(2.8),

$$\hat{a}_{ii} = -\frac{|u_i|}{|Du|u_i^3} \hat{h}_{ii} = -\frac{u_{ii}}{|Du|} = O(\mathcal{H}_\phi) + \eta_0 g \cdot O(1), \quad \forall j \in B,$$

and

$$\hat{a}_{jn}^2 = \left[ -\frac{|u_i|}{|Du|u_i^3} \frac{1}{\hat{W}} \hat{h}_{jn} \right]^2 \leq \hat{a}_{jj} \hat{a}_{nn} = O(\mathcal{H}_\phi) + \eta_0 g \cdot O(1), \quad \forall j \in B,$$

then we get

$$(5.17) \quad \hat{h}_{jn}^2 = O(\mathcal{H}_\phi) + \eta_0 g \cdot O(1), \quad \forall j \in B.$$

Following the proof of Lemma 3.1 in [12], we can get

$$\begin{aligned}
 &\sum_{\alpha,\beta=1}^n F^{\alpha\beta} \phi_{\alpha\beta} - \phi_t \\
 &= u_n^{-3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] \left[ -u_n^2 \left( \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta jj} - u_{jjt} \right) - 2u_n u_{nj} u_{jt} \right. \\
 &\quad \left. + 6u_n u_{nj} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta j} - 6u_{nj}^2 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta} \right] \\
 &\quad + 2u_n^{-3} \sum_{j \in B, i \in G} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] \sum_{\alpha,\beta=1}^n F^{\alpha\beta} \frac{1}{u_{ii}} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}] [u_n u_{ij\beta} - 2u_{i\beta} u_{jn}] \\
 (5.18) \quad &+ \eta_0 g \left[ -A^2 F^{nn} u_n^2 + A \cdot O(1) + O(1) \right] \\
 &- \frac{1}{\sigma_1^3(B)} \sum_{i \in B} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} [\sigma_1(B) \widetilde{a}_{ii,\alpha} - \widetilde{a}_{ii} \sum_{j \in B} \widetilde{a}_{jj,\alpha}] [\sigma_1(B) \widetilde{a}_{ii,\beta} - \widetilde{a}_{ii} \sum_{j \in B} \widetilde{a}_{jj,\beta}] \\
 &- \frac{1}{\sigma_1(B)} \sum_{i \neq j \in B} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} \widetilde{a}_{ij,\alpha} \widetilde{a}_{ij,\beta} + O(\mathcal{H}_\phi).
 \end{aligned}$$

From  $\hat{h}_{jn} = u_t^2 u_{jn} - u_n u_{jt} u_{jn}$ , we have

$$\begin{aligned}
 2u_n u_{jn} u_{jt} &= -\frac{1}{u_t} \left[ \left( \frac{\hat{h}_{jn}}{u_t} \right)^2 - (u_t u_{jn})^2 - (u_n u_{jt})^2 \right] \\
 (5.19) \quad &= O(\mathcal{H}_\phi) + \eta_0 g \cdot O(1) + \frac{1}{u_t} [(u_t u_{jn})^2 + (u_n u_{jt})^2].
 \end{aligned}$$



For each  $j \in B$ , differentiating equation (1.6) in  $x_j$  coordinate, it holds

$$\begin{aligned}
 u_{jji} &= \sum_{kl=1}^n F^{kl} u_{kljj} + \sum_{i=1}^n F^{u_i} u_{jji} + F^u u_{jj} \\
 &+ \sum_{klmn=1}^n F^{kl,mn} u_{klj} u_{mnj} + 2 \sum_{ikl=1}^n F^{kl,u_i} u_{klj} u_{ij} + 2 \sum_{kl=1}^n F^{kl,u} u_{klj} u_j \\
 &+ 2 \sum_{kl=1}^n F^{kl,x_j} u_{klj} + \sum_{ik=1}^n F^{u_i,u_k} u_{ij} u_{kj} + 2 \sum_{i=1}^n F^{u_i,u} u_{ij} u_j + 2 \sum_{i=1}^n F^{u_i,x_j} u_{ij} \\
 &+ F^{u,u} u_j^2 + 2F^{u,x_j} u_j + F^{x_j,x_j} \\
 &= \sum_{kl=1}^n F^{kl} u_{kljj} + 2 \frac{F^{u_n}}{u_n} u_{jn}^2 + \sum_{klmn=1}^n F^{kl,mn} u_{klj} u_{mnj} + 2 \sum_{kl=1}^n F^{kl,u_n} u_{klj} u_{jn} \\
 &+ 2 \sum_{kl=1}^n F^{kl,x_j} u_{klj} + F^{u_n,u_n} u_{jn}^2 + 2F^{u_n,x_j} u_{jn} + F^{x_j,x_j} \\
 &+ O(\mathcal{H}_\phi) + \eta_0 g[\cdot A \cdot O(1) + O(1)].
 \end{aligned} \tag{5.20}$$

Set

$$\begin{aligned}
 Q_j &= \sum_{klpq=1}^n F^{kl,pq} u_{klj} u_{pqj} u_n^2 + 2 \sum_{kl=1}^n F^{kl,u_n} u_{klj} u_{jn} u_n^2 + 2 \sum_{kl=1}^n F^{kl,x_j} u_{klj} u_n^2 + F^{u_n,u_n} u_{jn}^2 u_n^2 \\
 &+ 2F^{u_n,x_j} u_{jn} u_n^2 + F^{x_j,x_j} u_n^2 + 2F^{u_n} u_n u_{jn}^2 + 6u_n u_{jn} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{j\alpha\beta} - 6u_{jn}^2 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta} \\
 &+ 2 \sum_{i \in G} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} \frac{1}{u_{ii}} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}] [u_n u_{ij\beta} - 2u_{i\beta} u_{jn}],
 \end{aligned} \tag{5.21}$$

and denote

$$\begin{aligned}
 s &= \frac{1}{u_n} = \frac{1}{|\nabla u|}, A_{ij} = s u_{ij} = \frac{u_{ij}}{u_n}, \theta = (0, 0, \dots, 0, 1); \\
 X_{\alpha\beta} &= 2u_{\alpha\beta} u_{jn}, \quad \alpha \in B \text{ or } \beta \in B; \\
 X_{\alpha\beta} &= u_{\alpha\beta} u_n, \quad \text{otherwise}; \\
 Y &= u_{jn} u_n; \\
 Z_i &= \delta_{ij}, \quad i = 1, 2, \dots, n; \\
 \tilde{V} &= ((X_{\alpha\beta}), Y, (Z_i), 0) \in \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R};
 \end{aligned}$$

then we can get

$$X_{i\alpha} - 2A_{i\alpha} Y = 0, \quad i \in B.$$

So it yields

$$Q_j = Q^*(\tilde{V}, \tilde{V}), \tag{5.22}$$

where  $Q^*(\tilde{V}, \tilde{V})$  is defined in (2.25).

From (5.18), (5.20) and (5.21)), it yields

$$\begin{aligned}
 & \sum_{\alpha, \beta=1}^n F^{\alpha\beta} \phi_{\alpha\beta} - \phi_t \\
 = & u_n^{-3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] (Q_j - 2u_n u_{nj} u_{jt}) \\
 & + \eta_0 g \left[ -A^2 F^{nn} u_n^2 + AO(1) + O(1) \right] \\
 & - \frac{1}{\sigma_1^3(B)} \sum_{i \in B} \sum_{\alpha, \beta=1}^n F^{\alpha\beta} [\sigma_1(B) \tilde{a}_{ii, \alpha} - \tilde{a}_{ii} \sum_{j \in B} \tilde{a}_{jj, \alpha}] [\sigma_1(B) \tilde{a}_{ii, \beta} - \tilde{a}_{ii} \sum_{j \in B} \tilde{a}_{jj, \beta}] \\
 & - \frac{1}{\sigma_1(B)} \sum_{i \neq j \in B} \sum_{\alpha, \beta=1}^n F^{\alpha\beta} \tilde{a}_{ij, \alpha} \tilde{a}_{ij, \beta} + O(\mathcal{H}_\phi).
 \end{aligned} \tag{5.23}$$

From the structural condition (1.6) (i.e. Remark 2.8), it implies

$$Q^*(\tilde{V}, \tilde{V}) \leq 0.$$

so for  $j \in B$ ,

$$Q_j = Q^*(\tilde{V}, \tilde{V}) \leq 0. \tag{5.24}$$

Condition (1.9) implies

$$F^{nn} \geq \lambda > 0.$$

Set

$$V_{i\alpha} = \sigma_1(B) \tilde{a}_{ii, \alpha} - \tilde{a}_{ii} \sum_{j \in B} a_{jj, \alpha}.$$

Combining (5.19), (5.23) and (5.24),

$$\begin{aligned}
 F^{\alpha\beta} \phi_{\alpha\beta} \leq & C(\phi + \sum_{i, j \in B} |\nabla \tilde{a}_{ij}|) - \lambda \left[ \frac{\sum_{i \neq j \in B, \alpha=1}^n \tilde{a}_{ij, \alpha}^2}{\sigma_1(B)} + \frac{\sum_{i \in B, \alpha=1}^n V_{i\alpha}^2}{\sigma_1^3(B)} \right] \\
 & + \eta_0 g \left[ -A^2 F^{nn} u_n^2 + AO(1) + O(1) \right].
 \end{aligned} \tag{5.25}$$

By Lemma 3.3 in [2], for each  $M \geq 1$ , for any  $M \geq |\gamma_i| \geq \frac{1}{M}$ , there is a constant  $C$  depending only on  $n$  and  $M$  such that,  $\forall \alpha$ ,

$$\sum_{i, j \in B} |\tilde{a}_{ij, \alpha}| \leq C(1 + \frac{1}{\lambda^2})(\sigma_1(B) + |\sum_{i \in B} \gamma_i \tilde{a}_{ii, \alpha}|) + \frac{\lambda}{2} \left[ \frac{\sum_{i \neq j \in B} |\tilde{a}_{ij, \alpha}|^2}{\sigma_1(B)} + \frac{\sum_{i \in B} V_{i\alpha}^2}{\sigma_1^3(B)} \right]. \tag{5.26}$$

Taking  $\gamma_i = \sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}$  for each  $i \in B$ , the Newton-MacLaurine inequality implies

$$\sigma_l(G) + 1 \geq \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \geq \sigma_l(G), \quad \forall j \in B.$$

and

$$\phi_\alpha = \sum_{ij=1}^{n-1} \frac{\partial \phi}{\partial \tilde{a}_{ij}} \tilde{a}_{ij, \alpha} = \sum_{i \in B} \gamma_i \tilde{a}_{ii, \alpha} + O(\phi). \tag{5.27}$$

Therefore we conclude from (5.26) and (5.27) that  $\sum_{i,j \in B} |\nabla a_{ij}|$  can be controlled by the rest terms on the right hand side in (5.25) and  $\phi + |\nabla \phi|$ . So if we choose  $A > 0$  big, depending on  $\|F\|_{C^2}$ ,  $n$ ,  $\lambda$ ,  $\sigma$ ,  $\|u\|_{C^3}$ , then (5.6) holds, and the proof of Theorem 1.5 is complete.  $\square$

**5.2. Proof of the second part of Theorem 1.7.** Following the proof of Theorem 1.5, we can prove the corresponding results for the parabolic equations (1.12), (1.13) and (1.14) as follows.

**Theorem 5.1.** *Suppose  $u \in C^{3,1}(\Omega \times (0, T])$  is a spacetime quasiconcave solution to the initial boundary problem (1.6) for the parabolic equation (1.12)-(1.13) and satisfies (1.10). Then*

$$(5.28) \quad \kappa^{u(x,t)} \geq \min\{\kappa^0, \kappa^1 e^{-A}\} e^{Au(x,t)} \quad \text{in } \Omega \times [0, T],$$

for some universal constant  $A$  depending only on  $\|F\|_{C^2}$ ,  $n$ ,  $\lambda$ ,  $\sigma$ ,  $\|u\|_{C^3}$ .

**PROOF.** The proof is following the proof of Theorem 1.7, and the proofs of Theorem 3.3, Theorem 3.5 and Theorem 3.7, with some modifications. The difference is mainly from (5.12), (5.15)-(5.17), and all the others is similar. So we omit it.  $\square$

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